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Hermite and Hermite–Fejér Interpolation and Associated Product Integration Rules on the Real Line: The L_∞ Theory

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We investigate convergence in a weighted L_∞-norm of Hermite–Fejér and Hermite interpolation and related approximation processes, when the interpolation points are zeros of orthogonal polynomials associated with weights $W^2 = e^{-2Q}$ on the real line. For example, if $H_n(W^2, f, x)$ denotes the nth Hermite–Fejér interpolation polynomial for $W^2 = e^{-2Q}$ and the function f, then we show that

$$\lim_{n\to\infty} \{\sup_{x\in\mathbb{R}} |H_n(W^2, f, x) - f(x)|\; W^2(x)[1+|Q'(x)|]^{-\kappa}(1+|x|)^{-1}\} = 0,$$

under suitable conditions on f, W^2, and κ. The weights to which the results are applicable include $W^2(x) = \exp(-|x|^\alpha)$, $\alpha > 1$, or $W^2(x) = \exp(-\exp_k(|x|^\alpha))$, $\alpha > 1$, $k \geq 1$, where $\exp_k$ denotes the kth iterated exponential. Convergence of product integration rules induced by the various approximation processes is then deduced. Essentially the conclusion of the paper is that by damping the error in approximation of f by Hermite–Fejér or Hermite interpolation by a factor $[1+|Q'(x)|]^{-\kappa}(1+|x|)^{-1}$, which decays much more slowly than the weight W^2, we can ensure sup-norm convergence under quite general conditions.

1. INTRODUCTION

Let $W := e^{-Q}$, where $Q: \mathbb{R} \to \mathbb{R}$ is even, continuous, and of at least polynomial growth at infinity. Let $\mathscr{P}_n$ denote the set of real polynomials of degree $\leq n$. Form the nth orthonormal polynomial for W^2,

$$p_n(x) := p_n(W^2; x) := \gamma_n x^n + \cdots \in \mathscr{P}_n, \qquad \gamma_n = \gamma_n(W^2) > 0, \tag{1.1}$$

$n = 1, 2, 3, \ldots$, satisfying

$$\int_{-\infty}^{\infty} p_n(x)\, p_m(x)\, W^2(x)\, dx = \delta_{mn}. \tag{1.2}$$

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Let us order the zeros $x_{jn} = x_{jn}(W^2)$ of $p_n(x)$ so that

$$-\infty < x_{nn} < x_{n-1,n} < \cdots < x_{1n} < \infty. \tag{1.3}$$

We omit the W^2 from x_{jn} or $p_n(x)$ (and so on) unless confusion can arise.

In this paper, we study the convergence in a weighted L_∞ norm of *Hermite–Fejér* and *Hermite* interpolation, and other approximation processes, when $\{x_{jn}\}_{j=1}^n$ are chosen as the interpolation points. Recall that if $f: \mathbb{R} \to \mathbb{R}$, then the *nth Hermite–Fejér interpolation polynomial* is $H_n(W^2, f, \cdot) \in \mathscr{P}_{2n-1}$ satisfying

$$\begin{aligned} H_n(W^2, f, x_{jn}) &= f(x_{jn}) \\ H_n'(W^2, f, x_{jn}) &= 0, \end{aligned} \tag{1.4}$$

$1 \leqslant j \leqslant n$. The type of result we show is

$$\lim_{n\to\infty} \{\sup_{x\in\mathbb{R}} |H_n(W^2, f, x) - f(x)|\; W^2(x)[1 + |Q'(x)|]^{-\kappa}(1 + |x|)^{-1}\} = 0, \tag{1.5}$$

under suitable conditions on f, W^2, and κ, including a bound on the orthonormal polynomials. Here the damping factor $[1 + |Q'(x)|]^{-\kappa}(1 + |x|)^{-1}$ decays very slowly relative to the weight W^2.

The corresponding convergence question was treated in an L_1-setting in an earlier paper [19]. A brief survey of the topic was given there, so is omitted here.

To introduce the Hermite interpolation operator, and related approximation processes, we need more notation. Let $l_{jn} \in \mathscr{P}_{n-1}$, $1 \leqslant j \leqslant n$, be the *fundamental polynomials of Lagrange interpolation*, satisfying

$$l_{jn}(x_{kn}) = \delta_{jk}, \qquad 1 \leqslant j, k \leqslant n. \tag{1.6}$$

The *fundamental polynomials of Hermite interpolation* are then

$$h_{jn}(x) := \left\{1 - \frac{p_n''(x_{jn})}{p_n'(x_{jn})}(x - x_{jn})\right\} l_{jn}^2(x), \qquad 1 \leqslant j \leqslant n, \tag{1.7}$$

and

$$\hat{h}_{jn}(x) := (x - x_{jn})\, l_{jn}^2(x), \qquad 1 \leqslant j \leqslant n. \tag{1.8}$$

$H_n(W^2, f, x)$ admits the representation

$$H_n(W^2, f, x) = \sum_{j=1}^n f(x_{jn})\, h_{jn}(x). \tag{1.9}$$

If $f'(x_{jn})$ exists, $1 \leqslant j \leqslant n$, then the *Hermite* (or *osculatory*) *interpolation polynomial* is

$$\hat{H}_n(W^2, f, x) := \sum_{j=1}^{n} f(x_{jn})\, h_{jn}(x) + \sum_{j=1}^{n} f'(x_{jn})\, \hat{h}_{jn}(x). \tag{1.10}$$

It is characterized by the interpolatory conditions

$$\hat{H}_n^{(k)}(W^2, f, x_{jn}) = f^{(k)}(x_{jn}), \qquad k = 0, 1;\ 1 \leqslant j \leqslant n, \tag{1.11}$$

and by the condition $\hat{H}_n(W^2, f, \cdot) \in \mathscr{P}_{2n-1}$.

Both H_n and $\hat{H}_n$ are special cases of the operator

$$H_n^*(W^2, f, \{d_{ln}\}, x) := \sum_{j=1}^{n} f(x_{jn})\, h_{jn}(x) + \sum_{j=1}^{n} d_{jn} \hat{h}_{jn}(x), \tag{1.12}$$

satisfying

$$\begin{aligned} H_n^*(W^2, f, \{d_{ln}\}, x_{jn}) &= f(x_{jn}) \\ H_n^{*\prime}(W^2, f, \{d_{ln}\}, x_{jn}) &= d_{jn}, \end{aligned} \tag{1.13}$$

$1 \leqslant j \leqslant n$.

In several classical cases [28] and in those treated in this paper, the contribution to $h_{jn}(x)$ from $\{p_n''(x_{jn})/p_n'(x_{jn})(x - x_{jn})\}\, l_{jn}^2(x)$, is negligible. It is then natural to introduce, as did Grünwald [9], the very simple positive operator

$$Y_n(W^2, f, x) := \sum_{j=1}^{n} f(x_{jn})\, l_{jn}^2(x). \tag{1.14}$$

Each of the approximation processes above generates a *product quadrature rule*, involving approximation of

$$I[k; f] := \int_{-\infty}^{\infty} k(x)\, f(x)\, dx. \tag{1.15}$$

Here the kernel k is typically the "difficult" component of the integrand kf, with known types of singularity or oscillatory behaviour. The component f typically has "smooth" behaviour. The idea of the product quadrature rule is to approximate $I[k; f]$ by

$$I_n[k; f] := \int_{-\infty}^{\infty} k(x)\, H_n(W^2, f, x)\, dx \tag{1.16}$$

$$= \sum_{j=1}^{n} f(x_{jn}) \left(\int_{-\infty}^{\infty} k(x)\, h_{jn}(x)\, dx \right). \tag{1.17}$$

Analogous rules are generated by $\hat{H}_n$, H_n^*, and S_n: We use the notation

$$\hat{I}_n[k;f] := \int_{-\infty}^{\infty} k(x)\,\hat{H}_n(W^2, f, x)\,dx; \tag{1.18}$$

$$I_n^*[k;f] := \int_{-\infty}^{\infty} k(x)\,H_n^*(W^2, f, \{d_{ln}\}, x)\,dx; \tag{1.19}$$

and

$$J_n[k;f] := \int_{-\infty}^{\infty} k(x)\,Y_n(W^2, f, x)\,dx. \tag{1.20}$$

For a discussion of these rules, see [4, 19]. Under mild conditions on f and k, we shall prove, for example, that

$$\lim_{n\to\infty} I_n[k;f] = I[k;f]. \tag{1.21}$$

The paper is organized as follows: In Section 2, we introduce our class of weights, and state the main results. In Section 3, we estimate certain quadrature sums, using Markov–Posse–Stieltjes inequalities. That section can be read independently, and is possibly of independent interest. In Section 4, we present some technical estimates, mostly proved elsewhere. In Section 5, we establish some Markov–Bernstein inequalities, and in Section 6, we obtain Christoffel function estimates and deduce spacing results for $\{x_{jn}\}$. Finally, in Section 7, we prove the results of Section 2.

We close this section by introducing additional notation. Throughout, C, $C_1, C_2, ...,$ denote positive constants independent of n, x, and $P \in \mathscr{P}_n$. The same symbol does not necessarily denote the same constant in different occurrences. We write $C = C(W)$ to denote (for example) dependence on W and $C \neq C(n, x)$ to emphasise that C is independent of n and x.

We use $\sim$ in the following sense: If $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ are sequences, then

$$c_n \sim d_n$$

means that

$$C_1 \leqslant c_n/d_n \leqslant C_2, \qquad n \geqslant 1.$$

Similar notation is used for functions and sequences of functions. For real x, $\langle x \rangle$ denotes the greatest integer $\leqslant x$. Let $\mathscr{S} \subset \mathbb{R}$. A function $f: \mathscr{S} \to (0, \infty)$ is said to have *increasing tendency* if

$$x, y \in \mathscr{S} \qquad \text{and} \qquad x \leqslant y \text{ implies } f(x) \leqslant Cf(y).$$

Finally, some further notation involving orthogonal polynomials: Let $W=e^{-Q}$ as above. The *nth Christoffel function* is [25]

$$\lambda_n(W^2, x) := \inf_{P\in\mathscr{P}_{n-1}} \int_{-\infty}^{\infty} (PW)^2(t)\,dt/P^2(x) \tag{1.22}$$

$$= 1\Big/\sum_{j=0}^{n-1} p_j^2(W^2; x). \tag{1.23}$$

The *Christoffel numbers* are

$$\lambda_{jn} = \lambda_n(W^2, x_{jn}), \qquad 1 \leqslant j \leqslant n, \tag{1.24}$$

appearing in the *Gauss quadrature formula*

$$\int_{-\infty}^{\infty} P(x)\, W^2(x)\,dx = \sum_{j=1}^{n} \lambda_{jn} P(x_{jn}), \qquad P \in \mathscr{P}_{2n-1}. \tag{1.25}$$

If we set

$$\rho_n := \rho_n(W^2) := \gamma_{n-1}(W^2)/\gamma_n(W^2), \qquad n \geqslant 1, \tag{1.26}$$

then l_{jn} admits the representation [25]

$$l_{jn}(x) = \lambda_{jn}\rho_n p_{n-1}(x_{jn})\, p_n(x)/(x - x_{jn}), \qquad 1 \leqslant j \leqslant n. \tag{1.27}$$

2. Main Results

In our recent paper on convergence of Hermite–Fejér interpolation in the L_1-setting, we treated weights $W^2 = e^{-2Q}$, where Q is of polynomial, or of faster than polynomial growth at infinity. These are called respectively the *Freud* and *Erdös* cases. Here we also handle simultaneously Freud and Erdös weights, but have to assume slightly different hypotheses in the two cases:

Definition 2.1. We write $W \in \mathscr{W}_1$ if

(a) $W = e^{-Q}$, where $Q: \mathbb{R} \to \mathbb{R}$ is even, continuously differentiable, Q'' exists in $(0, \infty)$, and

$$Q(0) = 0. \tag{2.1}$$

(b) For $x \in (0, \infty)$,

$$Q'(x) > 0 \qquad \text{and} \qquad Q''(x) \geqslant 0. \tag{2.2}$$

(c) For $x \in (0, \infty)$, let

$$T(x) := (xQ'(x))'/Q'(x) = 1 + xQ''(x)/Q'(x). \tag{2.3}$$

Assume that for some $\eta > 0$,

$$T(x) \geqslant 1 + \eta, \qquad x \in (0, \infty). \tag{2.4}$$

Assume further that either

(i) T is bounded in $(0, \infty)$,

or

(ii) T is unbounded and increasing in $(0, \infty)$, satisfying there

$$T(x) \leqslant C[\log(2 + Q'(x))]^2. \tag{2.5}$$

The above conditions are slightly more restrictive than those in [19]. While (i) guarantees that Q is of polynomial growth at infinity, (ii) forces Q to be of faster than polynomial growth at infinity. In the latter case, (2.5) is a rather weak regularity condition: Under very general conditions on Q, it is satisfied for "most" x.

As examples of Q for which $W = e^{-Q} \in \mathscr{W}_1$, we mention

$$Q(x) := |x|^\beta, \qquad \beta > 1 \tag{2.6}$$

(this satisfies (i)) and

$$Q(x) := \exp_k(|x|^\beta) - \exp_k(0), \qquad \beta > 1, k \geqslant 1, \tag{2.7}$$

where $\exp_k := \exp(\exp(\dots\exp(\ )\dots))$ denotes the kth iterated exponential (this satisfies (ii)).

We also need to assume a bound on the orthonormal polynomials, and to this end, must define the *Mhaskar–Rahmanov–Saff number* $a_u = a_u(Q)$, $u > 0$. This is the positive root of the equation

$$u = \frac{2}{\pi} \int_0^1 \frac{a_u t Q'(a_u t)}{(1 - t^2)^{1/2}}\, dt, \qquad u > 0. \tag{2.8}$$

Since $sQ'(s)$ is positive and increasing for $s \in (0, \infty)$, with limits 0 and ∞ at 0 and ∞ respectively, a_u is uniquely defined. Moreover, a_u is increasing for $u \in (0, \infty)$, with limits 0 and ∞ at 0 and ∞ respectively. The significance of a_u is explored in [20, 22, 23].

DEFINITION 2.2. Let $W := e^{-Q} \in \mathscr{W}_1$. We write $W \in \mathscr{W}_2$, if there exists $\sigma := \sigma(W) > 0$, such that for $n \geqslant 1$,

$$\| p_n W[1 + |Q'|]^{-\sigma} \|_{L_\infty(\mathbb{R})} \leqslant C a_n^{-1/2}. \tag{2.9}$$

We remark that $\sigma=\sigma(W)>0$ exists, when, for example, Q is given by (2.6) with $\beta>1$, or by (2.7). This follows from asymptotics given for $p_n(W^2; x)$ in [14, 29]. For these Q, and for all those Q for which $W=e^{-Q}\in \mathscr{W}_1$ and for which orthogonal polynomial asymptotics were given in [14], one can choose any

$$\sigma \geqslant \frac{1}{2}\left(1+\frac{1}{\eta}\right), \tag{2.10}$$

where η is as in (2.4). This follows from the bounds in [14] and Nikolskii inequalities in [24, 27], and from the bound (4.5) below.

For the case where Q, is given by (2.6), with β a positive even integer, Bonan and Clark [2] showed that

$$\|p_n^2 W^2\|_{L_\infty(\mathbb{R})} \leqslant C n^{1/3-1/\beta}, \qquad n \geqslant 1. \tag{2.11}$$

This bound implies that in this case we can choose

$$\sigma \geqslant \tfrac{1}{6}\beta/(\beta-1) \qquad (\leqslant \tfrac{1}{3}). \tag{2.12}$$

Results of Bauldry [1] imply (2.11), and so allow (2.12), for more general weights.

While the assumption (2.9) does not yield the sharpest possible results, it allows tractable and close to sharp results, for all the weights for which orthogonal polynomial asymptotics were given in [14].

Following is our main theorem, establishing boundedness in weighted sup-norms, of $\{H_n\}$, $\{\hat{H}_n\}$, $\{Y_n\}$, and $\{H_n^*\}$:

THEOREM 2.3. *Let $W\in\mathscr{W}_2$ and $\sigma=\sigma(W)$. Let $f:\mathbb{R}\to\mathbb{R}$ satisfy, for some $\varepsilon>0$,*

$$A := \sup_{x\in\mathbb{R}} |f(x)|\ W^2(x)[1+|Q'(x)|]^{2\sigma+2+\varepsilon}(1+|x|)^2<\infty. \tag{2.13}$$

If T is bounded, let

$$\kappa>2\sigma+1, \tag{2.14a}$$

and if T is unbounded, let

$$\kappa>\max\{2\sigma+1, 4\sigma\}. \tag{2.14b}$$

Furthermore, let

$$V(x) := [1+|Q'(x)|]^{-\kappa}(1+|x|)^{-1}, \qquad x\in\mathbb{R}. \tag{2.15}$$

(a) *Then for* $n \geqslant 1$,

$$\| Y_n(W^2, f, \cdot)\, W^2 V\|_{L_\infty(\mathbb{R})} \leqslant C_1 A, \tag{2.16}$$

where $C_1 \neq C_1(n, f)$.

(b) *Further for* $n \geqslant 1$,

$$\|(Y_n(W^2, f, \cdot) - H_n(W^2, f, \cdot))\, W^2 V\|_{L_\infty(\mathbb{R})} \leqslant C_2 \left(\frac{a_n}{n}\right)^{1/2} A = o(1), \tag{2.17}$$

where $C_2 \neq C_2(n, f)$.

(c) *Suppose that* $\{d_{jn}\}_{j,n}$ *satisfy, for some* $\delta > 0$,

$$B := \sup_{\substack{n \geqslant 1 \\ 1 \leqslant j \leqslant n}} |d_{jn}|\, W^2(x_{jn})[1 + |Q'(x_{jn})|]^{2\sigma + 1 + \delta}(1 + |x_{jn}|)^2 < \infty. \tag{2.18}$$

Then for $n \geqslant 1$,

$$\|(Y_n(W^2, f, \cdot) - H_n^*(W^2, f, \{d_{ln}\}, \cdot))\, W^2 V\|_{L_\infty(\mathbb{R})} \leqslant C_3 \left(\frac{a_n}{n}\right)^{1/2} [A + B] = o(1), \tag{2.19}$$

where $C_3 \neq C_3(n, f, \{d_{ln}\})$.

(d) *Suppose that* f' *exists in* $\mathbb{R}$, *and for some* $\delta > 0$,

$$D := \sup_{x \in \mathbb{R}} |f'(x)|\, W^2(x)[1 + |Q'(x)|]^{2\sigma + 1 + \delta}(1 + |x|)^2 < \infty. \tag{2.20}$$

Then for $n \geqslant 1$,

$$\|(Y_n(W^2, f, \cdot) - \hat{H}_n(W^2, f, \cdot))\, W^2 V\|_{L_\infty(\mathbb{R})} \leqslant C_4 \left(\frac{a_n}{n}\right)^{1/2} [A + D] = o(1), \tag{2.21}$$

where $C_4 \neq C_4(n, f)$.

Remarks. (i) The message of (b)–(d) is that the operators H_n, H_n^*, and $\hat{H}_n$ are "equiconvergent" with the very simple positive operators

$$Y_n(W^2, f, x) := \sum_{j=1}^{n} f(x_{jn})\, l_{jn}^2(x), \qquad n \geqslant 1.$$

(ii) We note that the damping factor V decays extremely slowly in comparison to W^2. It is undoubtedly not the weakest possible damping factor for $W=e^{-Q}$, where Q is given by (2.6) and (2.7), but for the general class $\mathscr{W}_2$ (and given the present knowledge about orthogonal polynomials for weights on $\mathbb{R}$) is all that can be achieved. The value of κ and the powers of $(1+|Q'|)$ and $(1+|x|)$ can be improved for some of the statements above—see Theorems 7.5 and 7.6.

(iii) Even when T is unbounded, mild additional conditions allow us to assume (2.14a) rather than (2.14b). In particular, all the Erdös weights treated in [14] allow this choice, as the correct lower bounds are available for the Christoffel functions.

(iv) For the Freud case, $|Q'|$ is of polynomial growth, so $V(x)$, defined by (2.15), can be replaced by $(1+|x|)^{-\alpha}$ for suitable $\alpha>0$. However, this is not possible for the Erdös case. In both cases we could replace $V(x)$ by $(1+|Q'(x)|)^{-\beta}$, but again as $|Q'|$ can grow so much faster than any power of $|x|$, this would weaken the statements.

(v) We remark that the bound

$$\|(p_n W)(x)\, |1-(x/a_n)^2|^{1/4}\|_{L_\infty(\mathbb{R})} \leqslant C a_n^{-1/2},$$

has been established for the weight $W^2(x)=\exp(-x^{2m})$, m a positive integer [2]. Assuming such a bound, or more generally,

$$\|p_n W u_n\|_{L_\infty(\mathbb{R})} \leqslant C a_n^{-1/2},$$

allows us to prove analogous of Theorem 2.3, with V replaced by factors involving u_n. See [13, 14, 21] for related bounds.

The boundedness of the operators above, and density of the polynomials in suitably weighted spaces, yields:

COROLLARY 2.4. *In addition to the hypotheses of Theorem* 2.3, *suppose that f is continuous.*

(a) *Then*

$$\lim_{n\to\infty} \|(Y_n(W^2, f, \cdot)-f(\cdot))\, W^2 V\|_{L_\infty(\mathbb{R})}=0, \tag{2.22}$$

and

$$\lim_{n\to\infty} \|(H_n(W^2, f, \cdot)-f(\cdot))\, W^2 V\|_{L_\infty(\mathbb{R})}=0. \tag{2.23}$$

(b) *Assuming* (2.18), *we have*

$$\lim_{n\to\infty} \|(H_n^*(W^2, f, \{d_{ln}\}, \cdot)-f(\cdot))\, W^2 V\|_{L_\infty(\mathbb{R})}=0, \tag{2.24}$$

and assuming f' exists in $\mathbb{R}$ and satisfies (2.20), *we have*

$$\lim_{n\to\infty} \|(\hat{H}_n(W^2, f, \cdot) - f(\cdot))\, W^2 V\|_{L_\infty(\mathbb{R})} = 0. \tag{2.25}$$

For the special case of the Hermite weight, $W^2(x) = \exp(-x^2)$, (2.23) contains a substantial improvement of the results announced by Knoop [10] that $H_n(W^2, f, \cdot) \to f$ uniformly on compact subsets of $\mathbb{R}$, when f is of at most polynomial growth at infinity. To the best of my knowledge, Hermite–Fejér interpolation for weights on $\mathbb{R}$ has only been considered previously for the Hermite weight.

For the product integration rules $I_n, \hat{I}_n, I_n^*$, and J_n, defined by (1.16)–(1.20), we can prove:

COROLLARY 2.5. *Assume the hypotheses of Corollary* 2.4 *on f and W^2. Let $k: \mathbb{R} \to \mathbb{R}$ be measurable, and assume that*

$$\int_{-\infty}^{\infty} |k(x)|\, W^{-2}(x)\, V^{-1}(x)\, dx < \infty, \tag{2.26}$$

where V is given by (2.15).

(a) *Then*

$$\lim_{n\to\infty} J_n[k; f] = I[k; f] := \int_{-\infty}^{\infty} k(x) f(x)\, dx, \tag{2.27}$$

and

$$\lim_{n\to\infty} I_n[k; f] = I[k; f]. \tag{2.28}$$

(b) *Assuming* (2.18), *we have*

$$\lim_{n\to\infty} I_n^*[k; f] = I[k; f], \tag{2.29}$$

and assuming f' exists in $\mathbb{R}$ and satisfies (2.20), *we have*

$$\lim_{n\to\infty} \hat{I}_n[k; f] = I[k; f]. \tag{2.30}$$

3. QUADRATURE SUM ESTIMATES

In this section, we estimate quadrature sums of the form

$$\sum_{j\in\mathcal{S}} \lambda_{jn} W^{-2}(x_{jn})\, |x - x_{jn}|^{-p},$$

for suitable $\mathcal{S}$ and $p > 0$. The main result—Theorem 3.1—is of independent interest, so is proved in greater generality than the rest of this paper. The Markov–Posse–Stieltjes inequalities applied in a suitable form, enable us to estimate the above sums.

THEOREM 3.1. *Let* $W := e^{-Q}$, *where* $Q\colon \mathbb{R} \to \mathbb{R}$ *is continuous, even, and differentiable in* $(0, \infty)$. *Assume further that*

(a) $sQ'(s)$ *is positive and increasing in* $(0, \infty)$.

(b) *There exists* $B > 1$ *such that*

$$BsQ'(Bs) - sQ'(s) \geqslant 1, \; s \geqslant 1. \tag{3.1}$$

Let p *be a positive even integer, let* $n \geqslant 1$, *and* $x \in \mathbb{R}$. *Let* $x_{jn} = x_{jn}(W^2)$, $\lambda_{jn} = \lambda_{jn}(W^2)$, $1 \leqslant j \leqslant n$, *and choose*

$$l, m \in \{1, 2, ..., \langle n/2 \rangle\}, \tag{3.2}$$

such that

$$|x| > x_{ln}, \tag{3.3}$$

$$|x| < x_{m+1,n}. \tag{3.4}$$

Then

$$\sum_{j\,:\,|x_{jn}| \leqslant x_{l+1,n}} \lambda_{jn} W^{-2}(x_{jn})\, |x - x_{jn}|^{-p} \leqslant C_1(|x| - x_{ln})^{-p+1}, \tag{3.5}$$

and

$$\sum_{j\,:\,|x_{jn}| \geqslant x_{mn}} \lambda_{jn} W^{-2}(x_{jn})\, |x - x_{jn}|^{-p} \leqslant C_1(x_{m+1,n} - |x|)^{-p+1}, \tag{3.6}$$

where $C_1 \neq C_1(n, x, m, l)$ *but* $C_1 = C_1(W, p)$. *If we cannot choose* l *or* m *satisfying* (3.2) *to* (3.4) *then the corresponding statement is omitted.*

Remarks. (a) $Q(x) := |x|^\beta$, $\beta > 0$, satisfies the above hypotheses, as does $Q(x) := \exp_k(|x|^\beta)$, $\beta > 0$, $k \geqslant 1$.

(b) For arbitrary $p > 0$, we can estimate the sum in (3.5), see Lemmas 3.3 and 3.4.

(c) The estimation is possible largely because of the existence of an even entire function $G \sim W^{-2}$ constructed by Clunie and Kövari [3]. For finer results of this type, see [14, Chap. 6].

LEMMA 3.2. *Let* W *be as in Theorem* 3.1. *There exists an even entire function*

$$G(x) := \sum_{j=0}^{\infty} g_{2j} x^{2j}, \qquad g_{2j} \geqslant 0, \qquad j \geqslant 0, \tag{3.7}$$

such that for some $C_1, C_2 > 0$,

$$C_1 \leqslant G(x)\, W^2(x) \leqslant C_2, \qquad x \in \mathbb{R}. \tag{3.8}$$

Proof. In order to conform with the notation of [3], we define, for $r > 0$,

$$\phi(r) := \exp(2Q(r^{1/2}));$$
$$\psi(r) := r\phi'(r)/\phi(r) = r^{1/2} Q'(r^{1/2}).$$

Then ψ is a positive increasing function in $(0, \infty)$ and ϕ admits the representation

$$\phi(r) = \phi(1) \exp\left(\int_1^r \frac{\psi(s)}{s}\, ds\right), \qquad r \geqslant 1.$$

Furthermore, by (3.1), for $r \geqslant 1$,

$$\psi(B^2 r) - \psi(r) = B r^{1/2} Q'(B r^{1/2}) - r^{1/2} Q'(r^{1/2}) \geqslant 1.$$

By Theorem 4 of [3, pp. 19–20], there exists an entire

$$f(z) := \sum_{j=0}^{\infty} g_{2j} z^j, \qquad g_{2j} \geqslant 0, j \geqslant 0,$$

such that

$$\max_{|z|=r} |f(z)| = f(r) \sim \phi(r), \qquad r \geqslant 1.$$

Setting $G(x) := f(x^2)$, and increasing g_0 (if necessary) so as to make it positive, we obtain (3.8). ∎

The proof of Theorem 3.1 will be broken down into several steps.

LEMMA 3.3. *Let* W *be as in Theorem* 3.1. *Let* $p > 0$, *let* $n \geqslant 2$ *be a positive even integer, and let* $x \in \mathbb{R}$. *Choose* l *satisfying* (3.2) *and* (3.3). *Then*

$$\begin{aligned}\textstyle\sum_1 &:= \sum_{j:\, |x_{jn}| \leqslant x_{l+1,n}} \lambda_{jn} W^{-2}(x_{jn})\, |x - x_{jn}|^{-p} \\ &\leqslant C_1 \int_0^{x_{ln}} (|x| - t)^{-p}\, dt, \end{aligned} \tag{3.9}$$

where $C_1 \neq C_1(n, x, l, p)$. *The sum is taken as empty if the choice* (3.2), (3.3) *is not possible.*

Proof. Let $k := n/2$ and

$$U_1(t) := W^2(t^{1/2})/t^{1/2}, \qquad t \in (0, \infty),$$

and 0 otherwise. It is well known (cf. [6, p. 50; 11, p. 91]) that

$$\begin{aligned} p_n(W^2; t^{1/2}) &= p_k(U_1; t); \\ x_{jn}^2 = x_{jn}(W^2)^2 &= x_{jk}(U_1), \qquad 1 \leqslant j \leqslant k, \\ 2\lambda_{jn} = 2\lambda_{jn}(W^2) &= \lambda_{jk}(U_1), \qquad 1 \leqslant j \leqslant k. \end{aligned} \tag{3.10}$$

Then using the symmetry properties of the zeros and Christoffel numbers for the weight W^2, and using Lemma 3.2,

$$\begin{aligned} \sum\nolimits_1 &= \sum_{j: 0 < x_{jn} \leqslant x_{l+1,n}} \lambda_{jn} W^{-2}(x_{jn})[| \, |x| - x_{jn}|^{-p} + | \, |x| + x_{jn}|^{-p}] \\ &\leqslant C_1 \sum_{j: 0 < x_{jn} \leqslant x_{l+1,n}} \lambda_{jn} G(x_{jn}) \, 2 \, | \, |x| - x_{jn}|^{-p} \\ &\leqslant 2^{p+1} C_1 \, |x|^{-p} \sum_{j: 0 < x_{jn} \leqslant x_{l+1,n}} \lambda_{jn} G(x_{jn})[1 - (x_{jn}/|x|)^2]^{-p} \end{aligned}$$

(as $2^p(1 + x_{jn}/|x|)^{-p} \geqslant 1$)

$$= C_2 \, |x|^{-p} \sum_{j: 0 < x_{jn} \leqslant x_{l+1,n}} \lambda_{jn} H(x_{jn}^2) \, f(x_{jn}^2), \tag{3.11}$$

where

$$H(s) := G(s^{1/2}) = \sum_{j=0}^{\infty} g_{2j} \, s^j, \qquad s \in (0, \infty), \tag{3.12}$$

and

$$f(s) := (1 - s/x^2)^{-p} = \sum_{j=0}^{\infty} \left| \binom{-p}{j} \right| (s/x^2)^j, \qquad s \in [0, x^2). \tag{3.13}$$

Note that both H and f have non-negative Maclaurin series coefficients, and hence that Hf is absolutely monotone in $[0, x^2)$; that is,

$$(Hf)^{(j)}(s) \geqslant 0, s \in [0, x^2), \qquad j \geqslant 0.$$

Reexpressing (3.11) with the aid of (3.10) yields

$$\sum\nolimits_1 \leqslant C_3 \, |x|^{-p} \sum_{j=l+1}^{k} \lambda_{jk}(U_1) \, H(x_{jk}(U_1)) \, f(x_{jk}(U_1)).$$

We can now apply the classical Markov–Posse–Stieltjes inequality for the weight U_1 to deduce that

$$\sum\nolimits_1 \leqslant C_3 \, |x|^{-p} \int_0^{x_{lk}(U_1)} H(t) \, f(t) \, U_1(t) \, dt.$$

(See [6, pp. 32–33] for the Markov–Posse–Stieltjes inequality. However a clearer formulation appears in [18, p. 222, Lemma 3.2], but take account that there the zeros are ordered in increasing order.) Then

$$\sum\nolimits_1 \leqslant C_3\,|x|^{-p} \int_0^{x_{lk}^{1/2}(U_1)} H(s^2)\, f(s^2)\, W^2(s)\, 2\, ds$$
$$\leqslant C_4\,|x|^{-p} \int_0^{x_{ln}(W^2)} f(s^2)\, ds,$$

by (3.10) and by Lemma 3.2, which shows that

$$H(s^2)\, W^2(s) = G(s)\, W^2(s) \sim 1 \qquad \text{in } \mathbb{R}.$$

Finally,

$$|x|^{-p} f(s^2) = |x|^{-p}(1-(s/|x|)^2)^{-p}$$
$$\leqslant |x|^{-p}(1-s/|x|)^{-p} = (|x|-s)^{-p}. \quad \blacksquare$$

Next, we deal with n odd. This is a little more difficult.

LEMMA 3.4. *Let W be as in Theorem* 3.1. *Let $p>0$, let n be an odd positive integer, and let $x\in\mathbb{R}$. Choose l satisfying* (3.2) *and* (3.3). *Then* (3.9) *is valid for some $C_1 \neq C_1(n, x, l, p)$. The sum is taken as empty if the choice* (3.2), (3.3) *is not possible.*

Proof. Let $k := (n-1)/2$ and

$$U_2(t) := t^{1/2} W^2(t^{1/2}), \qquad t\in(0,\infty),$$

and 0 otherwise. It is known (cf. [6, p. 50], [11, pp. 89–90]) that

$$\begin{aligned} &p_n(W^2; t^{1/2})/t^{1/2} = p_k(U_2; t);\\ &x_{jn}^2 = x_{jn}(W^2)^2 = x_{jk}(U_2), \qquad 1\leqslant j\leqslant k,\\ &2\lambda_{jn} x_{jn}^2 = 2\lambda_{jn}(W^2)\, x_{jn}(W^2)^2 = \lambda_{jk}(U_2), \qquad 1\leqslant j\leqslant k. \end{aligned} \tag{3.14}$$

Then

$$\sum\nolimits_1 = |x|^{-p}\lambda_{k+1,n} W^{-2}(x_{k+1,n})$$
$$+ \sum_{j:\,0<x_{jn}\leqslant x_{l+1,n}} \lambda_{jn} W^{-2}(x_{jn})[|\,|x|-x_{jn}|^{-p} + |\,|x|+x_{jn}|^{-p}]$$
$$\leqslant |x|^{-p}\lambda_{k+1,n} W^{-2}(x_{k+1,n})$$
$$+ C_2\,|x|^{-p} \sum_{j:\,0<x_{jn}\leqslant x_{l+1,n}} \lambda_{jn} H(x_{jn}^2)\, f(x_{jn}^2),$$

exactly as at (3.11), where H and f are defined by (3.12) and (3.13). Note that we have used $x_{k+1,n}=0$. Now let

$$H_1(t) := (H(t)-H(0))/t = \sum_{j=1}^{\infty} g_{2j}\, t^{j-1}, \qquad t\in(0,\infty).$$

Then we have

$$\begin{aligned}\sum\nolimits_1 &\leqslant |x|^{-p}\lambda_{k+1,n} W^{-2}(x_{k+1,n})\\ &\quad + C_2\,|x|^{-p} \sum_{j:\,0<x_{jn}\leqslant x_{l+1,n}} \lambda_{jn} x_{jn}^2 H_1(x_{jn}^2)\, f(x_{jn}^2)\\ &\quad + C_2\,|x|^{-p} H(0) \sum_{j:\,0<x_{jn}\leqslant x_{l+1,n}} \lambda_{jn} f(x_{jn}^2)\\ &=: \sum\nolimits_{1,1} + \sum\nolimits_{1,2} + \sum\nolimits_{1,3},\end{aligned} \tag{3.15}$$

say. First, the classical Markov–Stieltjes inequality for W^2 yields

$$\begin{aligned}\sum\nolimits_{1,1} &= |x|^{-p}\lambda_{k+1,n} W^{-2}(0)\\ &\leqslant |x|^{-p}\left(\int_{x_{k+2,n}}^{x_{kn}} W^2(t)\,dt\right) W^{-2}(0)\\ &= 2\,|x|^{-p}\left(\int_0^{x_{kn}} W^2(t)\,dt\right) W^{-2}(0)\end{aligned}$$

(as $x_{k+2,n} = -x_{kn}$)

$$\leqslant 2\,|x|^{-p}\int_0^{x_{kn}} dt$$

(since $Q(t)\geqslant Q(0)$ in $(0,\infty)$)

$$\leqslant 2\int_0^{x_{kn}} (|x|-t)^{-p}\,dt \leqslant 2\int_0^{x_{ln}} (|x|-t)^{-p}\,dt. \tag{3.16}$$

Next, we handle the main term $\sum_{1,2}$: Using (3.14),

$$\begin{aligned}\sum\nolimits_{1,2} &= C_2\,|x|^{-p} \sum_{j:\,0<x_{jn}\leqslant x_{l+1,n}} \lambda_{jn} x_{jn}^2 H_1(x_{jn}^2)\, f(x_{jn}^2)\\ &= (C_2/2)\,|x|^{-p} \sum_{j=l+1}^{k} \lambda_{jk}(U_2)\, H_1(x_{jk}(U_2))\, f(x_{jk}(U_2))\\ &\leqslant (C_2/2)\,|x|^{-p}\int_0^{x_{lk}(U_2)} H_1(t)\, f(t)\, U_2(t)\,dt\end{aligned}$$

(by the Markov–Posse–Stieltjes inequality for U_2)

$$= C_2 |x|^{-p} \int_0^{x_{ln}(W^2)} H_1(s^2) f(s^2) W^2(s) s^2 \, ds$$

$$\leqslant C_2 |x|^{-p} \int_0^{x_{ln}(W^2)} H(s^2) f(s^2) W^2(s) \, ds$$

(for $s^2 H_1(s^2) = H(s^2) - H(0) \leqslant H(s^2)$)

$$\leqslant C_3 \int_0^{x_{ln}(W^2)} (|x| - s)^{-p} \, ds, \tag{3.17}$$

exactly as in the previous lemma. Finally, we estimate $\sum_{1,3}$. First, note that for $x_{jn} \in [0, |x|)$,

$$f(x_{jn}^2) = (1 - (x_{jn}/|x|)^2)^{-p} \leqslant (1 - x_{jn}/|x|)^{-p} = g(x_{jn}),$$

where

$$g(t) := (1 - t/|x|)^{-p}, \qquad t \in (-\infty, |x|).$$

Here g is absolutely monotone in $(-\infty, |x|)$, as is easily verified by successive differentiation. Then the Markov–Posse–Stieltjes inequality for W^2 [18, p. 222, Lemma 3.2(i), (ii)] yields

$$\sum_{1,3} = C_2 |x|^{-p} H(0) \sum_{j : 0 < x_{jn} \leqslant x_{l+1,n}} \lambda_{jn} f(x_{jn}^2)$$

$$\leqslant C_2 |x|^{-p} H(0) \sum_{j : 0 < x_{jn} \leqslant x_{l+1,n}} \lambda_{jn} \, g(x_{jn})$$

$$= C_2 |x|^{-p} H(0) \left[\sum_{j=l+1}^{n} - \sum_{j=k+1}^{n} \right] \lambda_{jn} \, g(x_{jn})$$

$$\leqslant C_2 |x|^{-p} H(0) \left[\int_{-\infty}^{x_{ln}} - \int_{-\infty}^{x_{k+1,n}} \right] g(t) \, W^2(t) \, dt$$

$$= C_2 |x|^{-p} H(0) \int_0^{x_{ln}} g(t) \, W^2(t) \, dt$$

$$\leqslant C_2 H(0) \, W^2(0) \int_0^{x_{ln}} (|x| - t)^{-p} \, dt,$$

as W is decreasing in $[0, \infty)$. Together with (3.15) to (3.17), this yields the result. ∎

Next, we handle the somewhat more problematic terms in (3.6). Here the summand can no longer be absolutely/completely monotone in the relevant range.

LEMMA 3.5. *Let W be as in Theorem 3.1. Let n and p be positive even integers and let $x \in \mathbb{R}$. Choose m satisfying* (3.2) *and* (3.4). *Then*

$$\sum\nolimits_2 := \sum_{j\,:\,|x_{jn}| \geqslant x_{mn}} \lambda_{jn} W^{-2}(x_{jn})\,|x - x_{jn}|^{-p}$$

$$\leqslant C_2 \int_{x_{m+1,n}}^{\infty} (t - |x|)^{-p}\,dt, \tag{3.18}$$

where $C_2 \neq C_2(n, x, m, p)$. The sum is taken as empty if the choice (3.2), (3.4) *is not possible.*

Proof. Let U_1 be as in Lemma 3.3 and let G be as in Lemma 3.2. Then

$$\sum\nolimits_2 := \sum_{j\,:\,x_{jn} \geqslant x_{mn}} \lambda_{jn} W^{-2}(x_{jn})[|\,|x| - x_{jn}|^{-p} + |\,|x| + x_{jn}|^{-p}]$$

$$\leqslant C_1 \sum_{j\,:\,x_{jn} \geqslant x_{mn}} \lambda_{jn} G(x_{jn})\, 2\, |\,|x| - x_{jn}|^{-p}$$

$$\leqslant 2^{p+1} C_1 \sum_{j\,:\,x_{jn} \geqslant x_{mn}} \lambda_{jn} G(x_{jn})(x_{jn}^2)^{-p/2}(1 - (|x|/x_{jn})^2)^{-p}$$

(as $2^p(1 + |x|/x_{jn})^{-p} \geqslant 1$)

$$= 2^{p+1} C_1 \sum_{j\,:\,x_{jn} \geqslant x_{mn}} \lambda_{jn} G(x_{jn})\, f_1(x_{jn}^2),$$

where

$$f_1(t) := t^{-p/2}(1 - x^2/t)^{-p} = \sum_{j=0}^{\infty} \left|\binom{-p}{j}\right| x^{2j} t^{-j-p/2}, \qquad t > x^2. \tag{3.19}$$

It is crucially important here that all these series coefficients be non-negative and that $j + p/2$ be an integer for $j \geqslant 0$. Then we can write

$$G(t)\, f_1(t^2) = \sum_{i=-\infty}^{\infty} \beta_i t^{2i}, \qquad t > x^2, \tag{3.20}$$

where $\beta_i \geqslant 0$ for all i. Then

$$\sum\nolimits_2 \leqslant 2^{p+1} C_1 \sum_{i=-\infty}^{\infty} \beta_i \left(\sum_{j\,:\,x_{jn} \geqslant x_{mn}} \lambda_{jn} x_{jn}^{2i} \right)$$

$$=: 2^{p+1} C_1 \sum_{i=-\infty}^{\infty} \beta_i \chi_i, \tag{3.21}$$

say. Of course, the interchanges are justified by non-negativeness of the series terms. Now we can use (3.10) to write

$$\chi_i = \sum_{j\,:\,x_{jn} \geqslant x_{mn}} \lambda_{jn} x_{jn}^{2i} = \tfrac{1}{2} \sum_{j=1}^{m} \lambda_{jk}(U_1)\, x_{jk}(U_1)^i. \tag{3.22}$$

Suppose first $i \geqslant 0$. Then the function $t \to t^i$ is absolutely monotone in $[0, \infty)$ in the sense that all its derivatives are non-negative. By the Markov–Posse–Stieltjes inequality for U_1 [18, p. 222, Lemma 3.2], [6, p. 92, Lemma 1.5]

$$\begin{aligned}\chi_i &= \tfrac{1}{2}\left(\sum_{j=1}^{n} - \sum_{j=m+1}^{n}\right) \lambda_{jk}(U_1)\, x_{jk}(U_1)^i \\ &\leqslant \tfrac{1}{2}\left(\int_0^\infty - \int_0^{x_{m+1,n}(U_1)}\right) t^i U_1(t)\, dt \\ &= \tfrac{1}{2} \int_{x_{m+1,n}(U_1)}^\infty t^i U_1(t)\, dt \\ &= \int_{x_{m+1,n}(W^2)}^\infty s^{2i} W^2(s)\, ds. \end{aligned} \tag{3.23}$$

Next, suppose $i < 0$. Then the function $t \to t^i$ is completely monotone in $(0, \infty)$ in the sense that

$$(-1)^j \left(\frac{d}{dt}\right)^j t^i > 0, \qquad t \in (0, \infty),\, j \geqslant 0.$$

Then the Markov–Posse–Stieltjes inequality for U_1 [18, p. 223, Lemma 3.3] yields

$$\chi_i \leqslant \frac{1}{2} \int_{x_{m+1,n}(U_1)}^\infty t^i U_1(t)\, dt = \int_{x_{m+1,n}(W^2)}^\infty s^{2i} W^2(s)\, ds.$$

Substituting (3.23), which we have shown valid for all $i = 0, \pm 1, \pm 2, \ldots$, into (3.21) yields

$$\begin{aligned}\Sigma_2 &\leqslant 2^{p+1} C_1 \int_{x_{m+1,n}(W^2)}^\infty \left(\sum_{i=-\infty}^{\infty} \beta_i s^{2i}\right) W^2(s)\, ds \\ &= 2^{p+1} C_1 \int_{x_{m+1,n}(W^2)}^\infty G(s)\, f_1(s^2)\, W^2(s)\, ds\end{aligned}$$

(by (3.20))

$$\leqslant C_2 \int_{x_{m+1,n}(W^2)}^{\infty} f_1(s^2)\, ds,$$

by Lemma 3.2. Finally,

$$f_1(s^2) = s^{-p}(1-(|x|/s)^2)^{-p} \leqslant s^{-p}(1-|x|/s)^{-p} = (s-|x|)^{-p}. \quad \blacksquare$$

Finally, we deal with the sum (3.6) for n odd.

LEMMA 3.6. *Let W be as in Theorem* 3.1. *Let n be an odd positive integer, p be an even positive integer, and let $x \in \mathbb{R}$. Choose m satisfying* (3.2) *and* (3.4). *Then* (3.18) *holds, where $C_2 \neq C_2(n, x, m, p)$. The sum is taken as empty if the choice* (3.2), (3.4) *is not possible.*

Proof. Let U_2 be as in Lemma 3.4. Let f_1 be as at (3.19). Exactly as in the previous lemma, we obtain (3.21). So we must estimate χ_i, $i=0, \pm 1, \pm 2, \ldots$. Using (3.14), we see that

$$\chi_i = \sum_{j:\, x_{jn} \geqslant x_{mn}} \lambda_{jn} x_{jn}^{2i} = \tfrac{1}{2} \sum_{j=1}^{m} \lambda_{jk}(U_2)\, x_{jk}(U_2)^{i-1}.$$

Proceeding exactly as in the previous lemma, we see that for all i,

$$\chi_i \leqslant \tfrac{1}{2} \int_{x_{m+1,n}(U_2)}^{\infty} t^{i-1} U_2(t)\, dt$$

$$= \int_{x_{m+1,n}(W^2)}^{\infty} s^{2i} W^2(s)\, ds.$$

We can then proceed as before to obtain (3.18). $\blacksquare$

Proof of Theorem 3.1. For n even, (3.5) follows from Lemma 3.3 and for n odd, from Lemma 3.4. For n even, (3.6) follows from Lemma 3.5, and for n odd, from Lemma 3.6. $\blacksquare$

4. TECHNICAL ESTIMATES

In this section, we list some technical estimates, mostly proved in other papers.

LEMMA 4.1. *Let $W \in \mathscr{W}_1$ and $\eta := \eta(W)$ be as in* (2.4).

(a) *For $x > 0$, $L \geqslant 1$,*

$$Q(x) \leqslant L^{-1} Q(Lx). \tag{4.1}$$

(b) $\exists C > 0$ *such that*

$$Q(a_u) \leqslant Cu, \qquad u \in (0, \infty). \tag{4.2}$$

(c) $\exists C_1, C_2, C_3$ *such that*

$$Q'(x) \geqslant C_1 x^{\eta}, \qquad x \geqslant C_3. \tag{4.3}$$

$$Q(x) \geqslant C_2 x^{1+\eta}, \qquad x \geqslant C_3. \tag{4.4}$$

(d)

$$a_u \leqslant C_4 u^{1/(1+\eta)}, \qquad u \geqslant C_5. \tag{4.5}$$

(e) *There exist* C_6, C_7, C_8 *such that for* $v \geqslant u \geqslant C_8$,

$$(1 + C_6/u)(v/u)^{1/(1+\eta)} \geqslant a_v/a_u \geqslant (v/u)^{C_7/(\log Q'(a_v))^2}. \tag{4.6}$$

Proof. (a) to (d) are Lemma 3.1 in [19]; (e) is Lemma 3.2 in [19]. Note that $\mathscr{W}_1$ is contained in the class $\mathscr{W}$ of [19]. ∎

We recall, for the reader's convenience, that "increasing tendency" was defined at the end of Section 1.

LEMMA 4.2. *Let* $W \in \mathscr{W}_1$ *and* $\eta := \eta(W)$ *be as in* (2.4).

(a) *For* $u \geqslant C_1$,

$$u \leqslant a_u Q'(a_u) \leqslant C_2 u T(a_u)^{1/2} \leqslant C_3 u \log u. \tag{4.7}$$

(b) *For* $u \geqslant C_4$,

$$Q'(a_u) \geqslant C_5 u^{\eta/(1+\eta)}. \tag{4.8}$$

(c) *T has increasing tendency in* $(0, \infty)$.

(d)

$$\max_{|x| \leqslant a_u} T(x) \leqslant C_6 (\log u)^2, \qquad u \geqslant C_7. \tag{4.9}$$

(e)

$$C_8/(uT(a_u)) \leqslant a_u'/a_u \leqslant 1/(u(1+\eta)), \qquad u \in (0, \infty). \tag{4.10}$$

(f)

$$a_{ru}/a_u \geqslant \exp\left(C_9 \frac{\log r}{T(a_{ru})}\right) \tag{4.11}$$

$$\geqslant 1 + C_9 (\log r)/T(a_{ru}) \tag{4.12}$$

$$\geqslant 1 + C_{10}/(\log u)^2, \tag{4.13}$$

$u \in (0, \infty)$, $r \in (1, \infty)$. *Here* $C_9 \neq C_9(r, u)$ *but* $C_{10} = C_{10}(r)$.

(g)

$$1 \leqslant a_{ru}/a_u \leqslant r^{1/(1+\eta)}, \qquad u \in (0, \infty), r \in (1, \infty). \tag{4.14}$$

Proof. (a) For the inequality $a_u Q'(a_u) \geqslant u$, see, for example, Lemma 3.3(b) in [19]. Next, when T is bounded, Lemma 3.1(c) in [12, p. 1071] shows (under less restrictive hypotheses) that

$$a_u Q'(a_u) \sim u,$$

and hence as T is bounded above and below,

$$a_u Q'(a_u) \sim u T(a_u)^{1/2}.$$

When T is unbounded, Lemma 2.2(c) in [15, p. 200] shows that

$$a_u Q'(a_u) \leqslant C u T(a_u)^{1/2}. \tag{4.15}$$

(Note that in [15], $\chi = T$ and we choose $j = 1$). Thus (4.15) holds whether T is bounded or unbounded. Finally, (2.5) yields the rightmost inequality in (4.7).

(b) This follows from the leftmost inequality in (4.7) and from (4.5).

(c) If T is unbounded, then we assumed it is increasing, and so trivially has increasing tendency. If T is bounded, then $T \sim 1$ in $(0, \infty)$, so the increasing tendency is again trivial.

(d) By (c), and by (2.5) and (4.7),

$$\max_{|x| \leqslant a_u} T(x) \leqslant C T(a_u) \leqslant C_1 (\log u)^2.$$

(e) Differentiating (2.8) with respect to u yields

$$1 = \frac{a_u'}{a_u} \frac{2}{\pi} \int_0^1 \frac{a_u t Q'(a_u t)\, T(a_u t)}{(1 - t^2)^{1/2}}\, dt.$$

Since

$$1 + \eta \leqslant T(a_u t) \leqslant C T(a_u), \qquad t \in (0, 1], u > 0,$$

the definition (2.8) of a_u yields

$$1 \leqslant C \left(\frac{a_u'}{a_u} \right) u T(a_u);$$

$$1 \geqslant \left(\frac{a_u'}{a_u} \right) u (1 + \eta).$$

Hence (4.10).

(f)

$$\begin{aligned} a_{ru}/a_u &= \exp\left(\int_u^{ru} a_t'/a_t\,dt\right) \\ &\geqslant \exp\left(C_9 \int_u^{ru} (tT(a_t))^{-1}\,dt\right) \qquad \text{(by (4.10))} \\ &\geqslant \exp\left(C_{11}\,T(a_{ru})^{-1} \int_u^{ru} t^{-1}\,dt\right) \\ &= \exp(C_{11}\,T(a_{ru})^{-1}\log r). \end{aligned}$$

Then (4.12) also follows and (4.9) yields (4.13).

(g) This is similar to (f): Use the upper bound in (4.10) rather than the lower bound. ∎

LEMMA 4.3. *Let $W \in \mathscr{W}_1$. There exists n_1 such that*

(i) $$\rho_n := \gamma_{n-1}(W^2)/\gamma_n(W^2) \leqslant a_{5n},\ n \geqslant n_1. \tag{4.16}$$

(ii) $$x_{1n} = x_{1n}(W^2) \leqslant a_{5n},\ n \geqslant n_1. \tag{4.17}$$

(iii) *For $n \geqslant 1$ and $1 \leqslant j \leqslant n$,*

$$|p_n''(x_{jn})/p_n'(x_{jn})| \leqslant C[1 + |Q'(x_{jn})|]. \tag{4.18}$$

Proof. For (i), (ii), see Corollary 4.4 in [19]. For (iii), we use (5.5) in Lemma 5.1 in [19] and Lemma 5.3 in [19] to deduce (with the notation there) that

$$\begin{aligned} |p_n''(x_{jn})/p_n'(x_{jn})| &= |2Q'(x_{jn}) + A_n'(x_{jn})/A_n(x_{jn})| \\ &\leqslant C[1 + |Q'(x_{jn})|]. \end{aligned}$$ ∎

5. MARKOV–BERNSTEIN INEQUALITIES

In this section, we present some L_∞ Markov–Bernstein inequalities. Recent work on this topic appears in [12, 17, 24, 26]. For fairly general Freud weights, it is known [12] that

$$\|P'W\|_{L_\infty(\mathbb{R})} + \|(PW)'\|_{L_\infty(\mathbb{R})} \leqslant C\frac{n}{a_n}\|PW\|_{L_\infty(\mathbb{R})}, \qquad P \in \mathscr{P}_n. \tag{5.1}$$

For fairly general Erdös weights W, it is known [15] that

$$\|P'W\|_{L_\infty(\mathbb{R})} + \|(PW)'\|_{L_\infty(\mathbb{R})} \leqslant C \frac{n}{a_n} T(a_n)^{1/2} \|PW\|_{L_\infty(\mathbb{R})}, \qquad P \in \mathscr{P}_n. \quad (5.2)$$

The unbounded factor $T(a_n)$ cannot be replaced by a more slowly growing factor [14].

Since we can only deal with n/a_n in the context of the paper, we prove inequalities in which $T(a_n)^{1/2}$ is replaced by a function of x, independent of n. The main result of this section is:

THEOREM 5.1. *Let $W := e^{-Q} \in \mathscr{W}_1$. Let $\beta \in \mathbb{R}$. Then for $n \geqslant 1$ and $P \in \mathscr{P}_n$,*

$$\|[|P'W| + |(PW)'|][1 + |Q'|]^\beta [\log(2 + |Q'|)]^{-2}\|_{L_\infty(\mathbb{R})}$$
$$\leqslant C \frac{n}{a_n} \|PW[1 + |Q'|]^\beta\|_{L_\infty(\mathbb{R})}. \quad (5.3)$$

Here $C \neq C(n, P)$. If T is bounded, then the factor $(\log(2 + |Q'|))^{-2}$ *may be omitted.*

We remark that when T is unbounded, stricter regularity assumptions on Q allow us to replace $(\log(2 + |Q'|))^{-2}$ by the "correct" factor $T^{-1/2}$. Our first step in proving Theorem 5.1 is an infinite–finite range inequality.

LEMMA 5.2. *Let $W := e^{-Q} \in \mathscr{W}_1$. Let $0 < p < \infty$, $\alpha \geqslant 0$ and β, $\Delta \in \mathbb{R}$. Then $\exists C \neq (n, P)$ such that for $n \geqslant 1$ and $P \in \mathscr{P}_n$,*

$$\|PW[1 + |Q'|]^\beta [\log(2 + |Q'|)]^\Delta\|_{L_p(\mathbb{R})}$$
$$\leqslant (1 + Cn^{-\alpha}) \|PW[1 + |Q'|]^\beta [\log(2 + |Q'|)]^\Delta\|_{L_p[-a_{4n}, a_{4n}]}. \quad (5.4)$$

Furthermore, for $n \geqslant 1$ and $P \in \mathscr{P}_n$,

$$\|PW[1 + |Q'|]^\beta [\log(2 + |Q'|)]^\Delta\|_{L_p(|t| \geqslant a_{4n})}$$
$$\leqslant Cn^{-\alpha} \|PW\|_{L_p[-a_{4n}, a_{4n}]}. \quad (5.5)$$

Proof. Under somewhat weaker conditions on Q than those for $\mathscr{W}_1$, it was shown in [19, Proof of Theorem 4.3] that if $\beta' \geqslant 0$,

$$\|PW[1 + |Q'|]^{\beta'}\|_{L_p(|t| \geqslant a_{4n})} \leqslant n^{-2\alpha} \|PW\|_{L_p(\mathbb{R})}, \qquad n \geqslant n_1, P \in \mathscr{P}_n.$$

Applying this with the special case $\beta' = 0$ to the right hand-side yields

$$\|PW[1 + |Q'|]^{\beta'}\|_{L_p(|t| \geqslant a_{4n})} \leqslant 2n^{-2\alpha} \|PW\|_{L_p[-a_{4n}, a_{4n}]}, \qquad n \geqslant n_2, P \in \mathscr{P}_n.$$

Since for any Δ, $\beta \in \mathbb{R}$, $\exists \beta' \geqslant 0$ such that

$$[1+|Q'|]^{\beta}[\log(2+|Q'|)]^{\Delta} \leqslant C_1[1+|Q'|]^{\beta'} \qquad \text{in } \mathbb{R},$$

we then obtain (5.5) at least for $n \geqslant n_2$. The remaining finitely many n can be treated by a compactness argument. To deduce (5.4) from (5.5), we note that

$$\begin{aligned}\|PW\|_{L_p[-a_{4n}, a_{4n}]} &\leqslant \|PW[1+|Q'|]^{\beta}[\log(2+|Q'|)]^{\Delta}\|_{L_p[-a_{4n}, a_{4n}]} \\ &\quad \times [1+Q'(a_{4n})]^{|\beta|}\,[\log(2+Q'(a_{4n}))]^{|\Delta|} \\ &\leqslant Cn^{\alpha/2}\,\|PW[1+|Q'|]^{\beta}\,[\log(2+|Q'|)]^{\Delta}\|_{L_p[-a_{4n}, a_{4n}]},\end{aligned}$$

if $\alpha/2 > |\beta|$, and we have used (4.7) to bound $Q'(a_{4n})$. Substituting this last inequality into (5.5) yields

$$\begin{aligned}&\|PW[1+|Q'|]^{\beta}[\log(2+|Q'|)]^{\Delta}\|_{L_p(|t| \geqslant a_{4n})} \\ &\quad \leqslant C_1 n^{-\alpha/2}\,\|PW[1+|Q'|]^{\beta}[\log(2+|Q'|)]^{\Delta}\|_{L_p[-a_{4n}, a_{4n}]}.\end{aligned}$$

Since α may be replaced by 2α and $a_{4n} \neq a_{4n}(\alpha)$, we obtain (5.4). ∎

We use [12] for Freud weights, and [15] for Erdös weights, in proving Theorem 5.1. First, Freud weights:

LEMMA 5.3. *Let $W \in \mathcal{W}_1$ and assume that T is bounded. Then $\exists C$ such that for $n \geqslant 1$ and $P \in \mathcal{P}_n$,*

$$\|\,|P'W| + |(PW)'|\,\|_{L_\infty(\mathbb{R})} \leqslant C\frac{n}{a_n}\,\|PW\|_{L_\infty(\mathbb{R})}. \tag{5.6}$$

Proof. By Theorem 1.1 in [12, p. 1066]

$$\|P'W\|_{L_\infty(\mathbb{R})} \leqslant \left(\int_1^{Cn} ds/Q^{[-1]}(s)\right)\|PW\|_{L_\infty(\mathbb{R})}, \tag{5.7}$$

$n \geqslant 1$, $P \in \mathcal{P}_n$. Here $Q^{[-1]}$ denotes the inverse function of Q. Also by Theorem 1.3 in [12, p. 1067] with $\eta = \frac{1}{2}$ there,

$$\|(PW)'\|_{L_\infty(|t| \geqslant a_{n/2})} \leqslant C\frac{n}{a_n}\,\|PW\|_{L_\infty(\mathbb{R})} \tag{5.8}$$

and

$$\|(PW)'\|_{L_\infty[-a_{n/2}, a_{n/2}]} \leqslant \left(\int_1^{Cn} ds/Q^{[-1]}(s)\right)\|PW\|_{L_\infty(\mathbb{R})}, \tag{5.9}$$

$n \geqslant 1$, $P \in \mathscr{P}_n$. It clearly suffices to show that

$$\theta_n := \int_1^{Cn} ds/Q^{[-1]}(s) \leqslant C_2 n/a_n. \tag{5.10}$$

Now it is shown in [12, p. 1071, Eq. (3.6)] that

$$a_n x Q'(a_n x) \sim Q(a_n x) \sim n, \tag{5.11}$$

uniformly for $x \in [a, b]$, any $0 < a < b < \infty$. Then for $n \geqslant 1$, $J \geqslant 1$, and some $C_3 \neq C_3(J, n)$,

$$Q(a_{Jn}) \geqslant C_3 Jn.$$

Together these imply that for some fixed integer $J > 0$, and with C as in (5.9),

$$Q(a_{Jn}) \geqslant Cn, \; n \geqslant n_1.$$

Then

$$\theta_n \leqslant \int_1^{Q(a_{Jn})} ds/Q^{[-1]}(s) = \int_{Q^{[-1]}(1)}^{a_{Jn}} Q'(t)/t \, dt.$$

Now for $0 < \varepsilon < \eta$,

$$\begin{aligned} \frac{d}{dt}(Q'(t)/t^{\varepsilon}) &= Q'(t)\, t^{-1-\varepsilon}(T(t) - 1 - \varepsilon) \\ &\geqslant (\eta - \varepsilon)\, Q'(t)\, t^{-1-\varepsilon} > 0, \qquad t > 0, \end{aligned}$$

so $Q'(t)/t^{\varepsilon}$ is increasing. Then

$$\begin{aligned} \theta_n &\leqslant Q'(a_{Jn})\, a_{Jn}^{-\varepsilon} \int_{Q^{[-1]}(1)}^{a_{Jn}} t^{-1+\varepsilon}\, dt \\ &\leqslant C_4 Q'(a_{Jn}) \leqslant C_5 n/a_n, \end{aligned}$$

by (5.11). Hence (5.10). ∎

LEMMA 5.4. *Let $W \in \mathscr{W}_1$ and assume that T is unbounded. Then $\exists C$ such that for $n \geqslant 1$ and $P \in \mathscr{P}_n$,*

$$\| [|P'W| + |(PW)'|][\log(2 + |Q'|)]^{-2} \|_{L_\infty(\mathbb{R})} \leqslant C \frac{n}{a_n} \|PW\|_{L_\infty(\mathbb{R})}. \tag{5.12}$$

Proof. Let us first assume that Q'' is continuous in $\mathbb{R}$. For $n \geqslant 1$ and $x \in [0, 1]$, let

$$\psi_n(x) := \int_{1/a_n}^{1} (1-s)^{-1/2} \frac{a_n x Q'(a_n x) - a_n s Q'(a_n s)}{a_n x - a_n s}\, ds$$

and

$$A_n^* := n^{-1} \int_{1/2}^{1} (1-s)^{-1/2} (a_n s)^2 Q''(a_n s)\, ds.$$

A result in [15, pp. 194–195] states that for $n \geqslant n_1$, $P \in \mathscr{P}_n$,

$$|(PW)'(x)| \leqslant C\, \|PW\|_{L_\infty(\mathbb{R})} \left(1 - \left|\frac{x}{a_n}\right|\right)^{-1} \int_{|x/a_n|}^{1} \psi_n(t)(1-t)^{1/2}\, dt,$$
$$|x| \leqslant a_n(1 - (nA_n^*)^{-2/3}). \tag{5.13}$$

Furthermore,

$$\|(PW)'\|_{L_\infty[-a_n/2, a_n/2]} \leqslant C \frac{n}{a_n} \|PW\|_{L_\infty(\mathbb{R})}. \tag{5.14}$$

Since [15, p. 200, Eq. (2.15)]

$$\lim_{n \to \infty} a_n Q'(a_n/2)/n = 0,$$

it follows that

$$\max\{|Q'(x)| : |x| \leqslant a_n/2\} = o(n/a_n),$$

so (5.14) also yields

$$\|[|P'W| + |(PW)'|]\|_{L_\infty[-a_n/2, a_n/2]} \leqslant C \frac{n}{a_n} \|PW\|_{L_\infty(\mathbb{R})}, \tag{5.15}$$

$P \in \mathscr{P}_n$, $n \geqslant n_1$.

We now deal with $|x| \geqslant a_n/2$. Now in [15, p. 208, Eq. (3.26)], it is shown that

$$\psi_n(t)(1-t)^{1/2} \sim \frac{n}{a_n} \mu_n(t) \text{ uniformly for } t \in [\tfrac{1}{2}, 1),$$

$n \geqslant n_1$. Here μ_n is a non-negative function in $[-1, 1]$ with [15, p. 205]

$$\int_{-1}^{1} \mu_n(t)\, dt = 1.$$

Hence for $|x| \in [a_n/2, a_n]$,

$$\int_{|x/a_n|}^{1} \psi_n(t)(1-t)^{1/2}\,dt \leqslant C_1 \frac{n}{a_n} \int_{|x/a_n|}^{1} \mu_n(t)\,dt \leqslant C_1 \frac{n}{a_n}.$$

Next, by definition of T, we can write

$$\begin{aligned} A_n^* &= n^{-1} \int_{1/2}^{1} a_n s Q'(a_n s)(1-s)^{-1/2}(T(a_n s)-1)\,ds \\ &\geqslant \eta n^{-1} \int_{1/2}^{1} a_n s Q'(a_n s)(1-s^2)^{-1/2}\,ds \text{ (by (2.4))} \\ &\geqslant \eta\pi/4, \end{aligned}$$

by definition of a_n and since $a_n s Q'(a_n s)(1-s^2)^{-1/2}$ is increasing in $(0, 1)$. Hence we can rewrite (5.13) in the form

$$|P'W|(x) + |(PW)'(x)| \leqslant \|PW\|_{L_\infty(\mathbb{R})} \left[|Q'(x)| + C_2 \frac{n}{a_n} \left(1 - \left|\frac{x}{a_n}\right|\right)^{-1} \right], \tag{5.16}$$

$n \geqslant n_1$, $P \in \mathscr{P}_n$, $a_n/2 \leqslant |x| \leqslant a_n(1 - C_3 n^{-2/3})$. In view of (4.13) in Lemma 4.2(f), we also have this valid for $a_n/2 \leqslant |x| \leqslant a_{n/2}$. Now for such x, write $|x| = a_u$, where $0 < u \leqslant n/2$. Then by (4.7) and (4.8),

$$\begin{aligned} |xQ'(x)| &\leqslant a_u Q'(a_u) \leqslant C_3 u(\log u) \\ &\leqslant C_4 u[\log(2 + Q'(a_u))] \leqslant C_4 \frac{n}{2}[\log(2 + |Q'(x)|)]. \end{aligned}$$

Since $|x| \geqslant a_n/2$, we obtain

$$|Q'(x)| \leqslant C_5 \frac{n}{a_n} \log(2 + |Q'(x)|), \tag{5.17}$$

$a_n/2 \leqslant |x| \leqslant a_{n/2}$. Also recalling our definition $|x| = a_u$,

$$\begin{aligned} \left(1 - \left|\frac{x}{a_n}\right|\right)^{-1} &= \left(1 - \frac{a_u}{a_n}\right)^{-1} \\ &\leqslant \left(1 - \frac{a_u}{a_{2u}}\right)^{-1} \qquad (\text{as } n \geqslant 2u) \\ &\leqslant C_6(\log u)^2 \qquad (\text{by (4.13)}) \\ &\leqslant C_7[\log(2 + Q'(a_u))]^2 = C_7[\log(2 + Q'(x))]^2, \end{aligned}$$

by (4.8). Together with (5.17) this enables us to write (5.16) in the form

$$|P'W|\,(x)+|(PW)'(x)|$$
$$\leqslant C_8\,\|PW\|_{L_\infty(\mathbb{R})}\frac{n}{a_n}\{[\log(2+|Q'(x)|)]+[\log(2+|Q'(x)|)]^2\},$$

$n\geqslant n_1$, $P\in\mathscr{P}_n$, $a_n/2\leqslant|x|\leqslant a_{n/2}$. Combined with (5.15), this yields

$$\|[|P'W|+|(PW)'|][\log(2+|Q'|)]^{-2}\|_{L_\infty[-a_{n/2},a_{n/2}]}$$
$$\leqslant C_9\,\|PW\|_{L_\infty(\mathbb{R})}\frac{n}{a_n},$$

$n\geqslant n_1$, $P\in\mathscr{P}_n$. Replacing n by $8n$,

$$\|[|P'W|+|(PW)'|][\log(2+|Q'|)]^{-2}\|_{L_\infty[-a_{4n},a_{4n}]}$$
$$\leqslant 8C_9\,\|PW\|_{L_\infty(\mathbb{R})}\frac{n}{a_n}, \tag{5.18}$$

$n\geqslant n_1$, $P\in\mathscr{P}_n$. Now by (5.5) of Lemma 5.2,

$$\|[|P'W|+|(PW)'|][\log(2+|Q'|)]^{-2}\|_{L_\infty(|t|\geqslant a_{4n})}$$
$$\leqslant\|[2\,|P'W|+|Q'|\,|PW|][\log(2+|Q'|)]^{-2}\|_{L_\infty(|t|\geqslant a_{4n})}$$
$$\leqslant C_{10}n^{-2}\,\|P'W\|_{L_\infty[-a_{4n},a_{4n}]}+C_{10}n^{-2}\,\|PW\|_{L_\infty[-a_{4n},a_{4n}]}$$

(by (5.18))

$$\leqslant C_{11}n^{-2}\frac{n}{a_n}[\log(2+Q'(a_{4n}))]^2\,\|PW\|_{L_\infty(\mathbb{R})}+C_{10}n^{-2}\,\|PW\|_{L_\infty[-a_{4n},a_{4n}]}$$
$$\leqslant C_{12}\,\|PW\|_{L_\infty(\mathbb{R})},$$

in view of (4.7). Together with (5.18), this establishes (5.12) for $n\geqslant n_1$. The remaining finitely many n can be treated by a compactness argument. Finally we note that we used the continuity of Q'' only in applying Lemma 3.2 in [15, p. 208]. Furthermore, (5.12) does not involve Q''. When Q'' is not continuous at 0, a straightforward argument yields (5.12) in the general case [15, pp. 221–222]. ∎

Next, we approximate the factor $[1+|Q'|]^\beta[\log(2+|Q'|)]^\Delta$ on $[-a_{\alpha n}, a_{\alpha n}]$:

LEMMA 5.5. *Let $W:=e^{-Q}\in\mathscr{W}_1$. Let $\beta,\Delta\in\mathbb{R}$ and $\alpha>0$. There exist polynomials R_n of degree at most $o(n)$, $n\to\infty$, such that for $n\geqslant 1$, we have in $[-a_{\alpha n}, a_{\alpha n}]$*

$$R_n\sim[1+|Q'|]^\beta[\log(2+|Q'|)]^\Delta \tag{5.19}$$

and

$$|R_n'| \leqslant C(\log(n+1))^3[1+|Q'|]^{\beta}[\log(2+|Q'|)]^{\Delta}. \tag{5.20}$$

Proof. We note first that we may assume Q'' is continuous at 0, since we are only looking for $\sim$ in (5.19): If not, modify Q in $[-1, 1]$ to obtain a twice differentiable Q_1 there. Now in $\mathbb{R}$,

$$\begin{aligned}\phi &:= [1+|Q'|]^{\beta}[\log(2+|Q'|)]^{\Delta}\\ &\sim [1+Q'^2]^{\beta/2}[\log(2+Q'^2)]^{\Delta} = \exp(\psi),\end{aligned} \tag{5.21}$$

where

$$\psi := \frac{\beta}{2}\log(1+Q'^2) + \Delta \log\log(2+Q'^2). \tag{5.22}$$

Let

$$f_n(t) := \psi(a_{2\alpha n} t), \qquad t \in [-1, 1], n \geqslant 1. \tag{5.23}$$

Then by (4.7),

$$\|f_n\|_{L_\infty[-1,1]} \leqslant C_1 \log(1+Q'(a_{2\alpha n})^2) \leqslant C_2 \log(n+1). \tag{5.24}$$

Also

$$\begin{aligned}|f_n'(t)| &= \left| a_{2\alpha n}\, \beta \frac{(Q'Q'')(a_{2\alpha n} t)}{1+Q'(a_{2\alpha n} t)^2}\right.\\ &\quad \left. + a_{2\alpha n}\, \Delta \frac{2(Q''Q')(a_{2\alpha n} t)}{(\log(2+Q'(a_{2\alpha n} t)^2))(2+Q'(a_{2\alpha n t})^2)}\right|\\ &\leqslant C_3 \frac{a_{2\alpha n} Q''(a_{2\alpha n} t)}{(1+Q'(a_{2\alpha n} t)^2)^{1/2}}.\end{aligned}$$

Now for $1 \leqslant |s| \leqslant a_{2\alpha n}$,

$$\begin{aligned}|Q''(s)/Q'(s)| &= |T(s)-1|/|s|\\ &\leqslant \|T\|_{L_\infty[-a_{2\alpha n}, a_{2\alpha n}]} \leqslant C_4(\log(n+1))^2.\end{aligned}$$

Then (4.14) and the continuity of Q'' ensures that

$$\|f_n'\|_{L_\infty[-1,1]} \leqslant C_5 a_n (\log(n+1))^2, \qquad n \geqslant 1. \tag{5.25}$$

Let φ_n denote the polynomial of degree $\langle C_5 a_n (\log(n+1))^2 \rangle$ of best uniform approximation to f_n on $[-1, 1]$. By Jackson's theorem on approximation by polynomials,

$$\|f_n - \varphi_n\|_{L_\infty[-1,1]} \leqslant C_6 \|f_n'\|_{L_\infty[-1,1]}/\text{degree}(\varphi_n) \leqslant C_7. \tag{5.26}$$

Now by standard results on derivatives of polynomials of best approximation [5, p. 84, pp. 10–11],

$$\begin{aligned} \max_{t\in[-1,1]} (1-t^2)^{1/2} |\varphi_n'(t)| &\leqslant C_8 \max_{t\in[-1,1]} (1-t^2)^{1/2} |f_n'(t)| \\ &\leqslant C_9 a_n (\log(n+1))^2. \end{aligned}$$

Then for $|t| \leqslant a_{\alpha n}/a_{2\alpha n}$,

$$\begin{aligned} |\varphi_n'(t)| &\leqslant C_9 a_n (\log(n+1))^2 (1-(a_{\alpha n}/a_{2\alpha n})^2)^{-1/2} \\ &\leqslant C_{10} a_n (\log(n+1))^3, \end{aligned} \tag{5.27}$$

by (4.13). Now let $\tau_k(u)$ denote the $(k+1)$th partial sum of the Maclaurin series of e^u, $k \geqslant 1$. It is well known that

$$C_{11} \leqslant \tau_k(u)\, e^{-u} \leqslant C_{12}, \qquad |u| \leqslant C_{13} k,\ k \geqslant 1. \tag{5.28}$$

Let us note from (5.24) and (5.26) that

$$\|\varphi_n\|_{L_\infty[-1,1]} \leqslant C_{14} \log(n+1).$$

Then we can choose $k = k(n) = O(\log(n+1))$, such that if

$$R_n(u) := \tau_{k(n)}(\varphi_n(u/a_{2\alpha n})),$$

then in view of (5.28), for $u \in [-a_{\alpha n}, a_{\alpha n}]$,

$$\begin{aligned} R_n(u) &\sim \exp(\varphi_n(u/a_{2\alpha n})) \\ &\sim \exp(f_n(u/a_{2\alpha n})) = \exp(\psi(u)) \sim \phi(u), \end{aligned}$$

by (5.26) and (5.23). Furthermore, for $u \in [-a_{\alpha n}, a_{\alpha n}]$,

$$\begin{aligned} |R_n'(u)| &= |\tau_{k(n)}'(\varphi_n(u/a_{2\alpha n}))|\ |\varphi_n'(u/a_{2\alpha n})\, a_{2\alpha n}^{-1}| \\ &= |\tau_{k(n)-1}(\varphi_n(u/a_{2\alpha n}))|\ |\varphi_n'(u/a_{2\alpha n})\, a_{2\alpha n}^{-1}| \\ &\sim \exp(\varphi_n(u/a_{2\alpha n}))\ |\varphi_n'(u/a_{2\alpha n})\, a_{2\alpha n}^{-1}| \\ &\leqslant C_{15} \exp(\psi(u)) \frac{a_n}{a_{2\alpha n}} (\log(n+1))^3 \qquad \text{(by (5.27))} \\ &\leqslant C_{16} \phi(u) (\log(n+1))^3, \end{aligned}$$

by (5.21) and (4.14). So we have (5.19) and (5.20).

Finally, the degree of R_n is $k(n)\langle C_5 a_n(\log(n+1))^2\rangle$, which is $O(a_n(\log(n+1))^3)$ and hence $o(n)$ by (4.5). ∎

Proof of Theorem 5.1. In view of Lemma 5.2 (cf. the proof of Lemma 5.4), it suffices to prove that

$$\|[|P'W|+|(PW)'|][1+|Q'|]^\beta[\log(2+|Q'|)]^{-2}\|_{L_\infty[-a_{4n},a_{4n}]}$$
$$\leqslant C_1\frac{n}{a_n}\|PW[1+|Q'|]^\beta\|_{L_\infty(\mathbb{R})}, \qquad P\in\mathscr{P}_n,\ n\geqslant 1. \tag{5.29}$$

So let $R_n\in\mathscr{P}_n$, $n\geqslant n_1$, be the polynomials of Lemma 5.5 with $\Delta=0$ and $\alpha=4$. Then in $[-a_{4n},a_{4n}]$ for $P\in\mathscr{P}_n$,

$$\begin{aligned}&[|P'W|+|(PW)'|][1+|Q'|]^\beta\\&\quad\sim[|P'W|+|(PW)'|]R_n\\&\quad=|(PR_n)'W-PR_n'W|+|(PR_nW)'-R_n'PW|\\&\quad\leqslant|(PR_n)'W|+|(PR_nW)'|+2|R_n'|\,|PW|\\&\quad\leqslant C\frac{2n}{a_{2n}}\|PR_nW\|_{L_\infty(\mathbb{R})}[\log(2+|Q'|)]^2+2|R_n'|\,|PW|,\end{aligned}$$

by Lemmas 5.3 and 5.4. Of course if T is bounded, Lemma 5.3 shows that $[\log(2+|Q'|)]^2$ can be omitted. Since $PR_n\in\mathscr{P}_{2n}$ [22],

$$\begin{aligned}\|PR_nW\|_{L_\infty(\mathbb{R})}&=\|PR_nW\|_{L_\infty[-a_{2n},a_{2n}]}\\&\sim\|PW[1+|Q'|]^\beta\|_{L_\infty[-a_{2n},a_{2n}]}.\end{aligned}$$

Then using the bound for R_n' from Lemma 5.5, we have in $[-a_{4n},a_{4n}]$ for $P\in\mathscr{P}_n$,

$$\begin{aligned}&[|P'W|+|(PW)'|][1+|Q'|]^\beta\\&\quad\leqslant C\frac{n}{a_n}\|PW[1+|Q'|]^\beta\|_{L_\infty[-a_{2n},a_{2n}]}[\log(2+|Q'|)]^2\\&\qquad+C(\log(n+1))^3\,|PW|\,[1+|Q'|]^\beta.\end{aligned}$$

Since

$$n/a_n\geqslant C_1(\log(n+1))^3, \qquad n\geqslant 1,$$

we have (5.29) and the theorem. ∎

6. Estimates for Christoffel Functions, and Consequences

In this section, we obtain upper and lower bounds for Christoffel functions, and apply these to spacing of the zeros of the orthogonal polynomials. The following lemma was proved in a weaker form in [19].

Lemma 6.1. *Let* $W \in \mathcal{W}_1$ *and* $\eta := \eta(W)$ *be as in* (2.4). *Let*

$$\beta := (1+\eta)/(2\eta). \tag{6.1}$$

Then $\exists \varepsilon \in (0, 1)$, $n_1 \geqslant 1$ *and* C, *such that if*

$$m := m(n) := \varepsilon n / T(a_n)^\beta, \; n \geqslant n_1, \tag{6.2}$$

we have

$$\lambda_n(W^2, x)\, W^{-2}(x) \leqslant C \frac{a_n}{n} \left[\frac{1}{n} + \left(1 - \left(\frac{x}{a_{4n}} \right)^2 \right)^{1/2} \right], \tag{6.3}$$

for

$$n \geqslant n_1 \text{ and } |x| \leqslant a_{m(n)}. \tag{6.4}$$

Proof. We adopt the old method of Freud [8]. By Lemma 5.2, for $n \geqslant n_1$ say,

$$\lambda_n(W^2, x)\, W^{-2}(x) \leqslant 2 \inf_{P \in \mathcal{P}_{n-1}} \int_{-a_{4n}}^{a_{4n}} (PW)^2(t)\, dt/(PW)^2(x). \tag{6.5}$$

Define for a fixed x the linear polynomial in t,

$$\psi_x(t) := Q(x) + (t-x)\, Q'(x).$$

If Q'' exists throughout $\mathbb{R}$, we see that by convexity,

$$\psi_x(t) - Q(t) = -\tfrac{1}{2} Q''(\xi)(t-x)^2 \leqslant 0, \qquad t \in \mathbb{R}.$$

As the left-hand side does not involve Q'', a continuity argument establishes this even when $Q''(0)$ does not exist. Hence

$$\exp(\psi_x(t))\, W(t) \leqslant 1, \qquad t \in \mathbb{R}. \tag{6.6}$$

Next, for $|t| \leqslant a_{4n}$ and $|x| \leqslant a_{m(n)}$,

$$|\psi_x(t)| \leqslant Q(a_{m(n)}) + 2 \frac{a_{4n}}{a_{m(n)}} a_{m(n)}\, Q'(a_{m(n)})$$
$$\leqslant C_1 m(n) + C_2 (n/m(n))^{1/(1+\eta)} m(n)\, T(a_{m(n)})^{1/2}$$

(by (4.2), (4.6) and (4.7))

$$\leqslant C_3 \varepsilon n + C_3 \varepsilon^{-1/(1+\eta)+1} T(a_n)^{\beta/(1+\eta)} nT(a_n)^{1/2-\beta}$$
$$= C_3 n[\varepsilon + \varepsilon^{\eta/(1+\eta)}], \tag{6.7}$$

by the choice (6.1) of β. Here $C_3 \neq C_3(\varepsilon, n, x, t)$.

Next, as in the proof of Lemma 5.5, let $\tau_k(u)$ denote the $(k+1)$th partial sum of e^u. Define

$$S_{x,n}(t) := \tau_{\langle n/2 \rangle}(\psi_x(t)).$$

From (5.28) and (6.7), if ε is small enough,

$$S_{x,n}(t) \sim \exp(\psi_x(t)), \qquad |t| \leqslant a_{4n}, |x| \leqslant a_{m(n)}.$$

Then by (6.6),

$$0 < S_{x,n}(t)\, W(t) \leqslant C_4, \qquad |t| \leqslant a_{4n}, |x| \leqslant a_{m(n)}. \tag{6.8}$$

Furthermore,

$$S_{x,n}(x)\, W(x) \sim \exp(\psi_x(x))\, W(x) = 1, \qquad |x| \leqslant a_{m(n)}. \tag{6.9}$$

Substituting $P(t) := S_{x,n}(t)\, R(t)$ in (6.5), where $R \in \mathscr{P}_{\langle n/2 \rangle - 1}$ is arbitrary, yields

$$\lambda_n(W^2, x)\, W^{-2}(x)$$
$$\leqslant 2 \inf_{R \in \mathscr{P}_{\langle n/2 \rangle - 1}} \int_{-a_{4n}}^{a_{4n}} (RS_{x,n} W)^2(t)\, dt / (RS_{x,n} W)^2(x)$$
$$\leqslant C_5 \inf_{R \in \mathscr{P}_{\langle n/2 \rangle - 1}} \int_{-a_{4n}}^{a_{4n}} R^2(t)\, dt / R^2(x),$$
$$\leqslant C_5 a_{4n} \lambda_{\langle n/2 \rangle}(w, x/a_{4n}),$$

where w is the classical Legendre weight on $[-1, 1]$. By classical estimates [30],

$$\lambda_l(w; s) \leqslant C_6 \frac{1}{l} \left[\frac{1}{l} + (1 - s^2)^{1/2} \right], \qquad s \in [-1, 1], l \geqslant 1.$$

Hence the result. ∎

LEMMA 6.2. *Let $W \in \mathscr{W}_1$. For $n \geqslant 1$,*

$$\sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn})(2 + x_{jn}^2)^{-1/2} (\log(2 + x_{jn}^2))^{-2} \leqslant C. \tag{6.10}$$

Proof. See [19, Lemma 6.2]. ∎

LEMMA 6.3. *Let $W \in \mathscr{W}_1$ and $\eta := \eta(W)$ be as in* (2.4). *Define β and $m(n)$ as in* (6.1) *and* (6.2). *Then*

$$|x_{jn}|, |x_{j+1,n}| \leqslant a_{m(n)} \tag{6.11}$$

implies

$$|x_{jn} - x_{j+1,n}| \leqslant C \frac{a_n}{n} \left(\frac{1}{n} + \left(1 - \left[\frac{\min\{|x_{jn}|, |x_{j+1,n}|\}}{a_{4n}} \right]^2 \right)^{1/2} \right). \tag{6.12}$$

Proof. We use the method of Freud [7, pp. 293–294]. Choose $D, E \in \mathbb{R}$ such that

$$\exp(Dx_{jn} + E)\, W^2(x_{jn}) = 1 = \exp(Dx_{j+1,n} + E)\, W^2(x_{j+1,n}).$$

By convexity of Q,

$$\exp(Dt + E)\, W^2(t) = \exp(Dt + E - 2Q(t)) \geqslant 1, \qquad t \in [x_{j+1,n}, x_{jn}]. \tag{6.13}$$

Also then by the Markov–Posse–Stieltjes inequality,

$$\begin{aligned} x_{jn} - x_{j+1,n} &\leqslant \int_{x_{j+1,n}}^{x_{jn}} \exp(Dt + E)\, W^2(t)\, dt \\ &\leqslant \lambda_{jn} \exp(Dx_{jn} + E) + \lambda_{j+1,n} \exp(Dx_{j+1,n} + E) \\ &= \lambda_{jn} W^{-2}(x_{jn}) + \lambda_{j+1,n} W^{-2}(x_{j+1,n}). \end{aligned}$$

Now apply the bounds of Lemma 6.1. ∎

LEMMA 6.4. *Let $W \in \mathscr{W}_1$ and $\eta := \eta(W)$ be as in* (2.4). *Let*

$$\Delta := (1 + \eta)/\eta. \tag{6.14}$$

Let ε_0 be small enough, and let

$$l := l(n) := \varepsilon_0 n / T(a_n)^{\Delta}, \qquad n \geqslant 1. \tag{6.15}$$

Then $\exists n_1$ such that for $n \geqslant n_1$, $p_n(x) = p_n(W^2; x)$ has at least one zero in $J_n := [a_{l(n)/2}, a_{l(n)}]$.

Proof. We use an argument of W. Hahn as adapted by Freud [7]. Suppose on the contrary that p_n has no zeros in J_n. Let

$$\psi_n(x) := T^2_{\langle n/2 \rangle - 1}(1 + (a_{l/2} - x)(x - a_l)/(4a_{5n}^2)).$$

Then for $x \in [-a_{5n}, a_{5n}] \backslash J_n$ and $n \geqslant n_1$,

$$0 \leqslant 1 + (a_{l/2} - x)(x - a_l)/(4a_{5n}^2) \leqslant 1,$$

so $0 \leqslant \psi_n(x) \leqslant 1$. By Lemma 4.3(ii) and our hypothesis, all zeros lie in $[-a_{5n}, a_{5n}] \backslash J_n$ for $n \geqslant n_1$, so

$$\sum_{j=1}^{n} \lambda_{jn} \psi_n(x_{jn}) \leqslant \sum_{j=1}^{n} \lambda_{jn} = \int_{-\infty}^{\infty} W^2(t)\, dt. \tag{6.16}$$

Also, by the Gauss quadrature formula,

$$\sum_{j=1}^{n} \lambda_{jn} \psi_n(x_{jn}) = \int_{-\infty}^{\infty} \psi_n(t)\, W^2(t)\, dt \geqslant \int_{K_n} \psi_n(t)\, W^2(t)\, dt, \tag{6.17}$$

where $K_n := [a_{2l/3}, a_{3l/4}]$. Now for $t \in K_n$,

$$\begin{aligned} 1 + (a_l - t)(t - a_{l/2})/(4a_{5n}^2) &\geqslant 1 + (a_l - a_{3l}/4)(a_{2l/3} - a_{l/2})/(4a_{5n}^2) \\ &\geqslant 1 + C_1 (a_{l/2}/a_{5n})^2 T(a_n)^{-2} \end{aligned}$$

(by (4.12) and as $n \geqslant l(n)$, $n \geqslant n_1$)

$$\geqslant 1 + C_2 \varepsilon_0^{2/(1+\eta)} T(a_n)^{-2\Delta/(1+\eta)-2},$$

(by (4.6) and the definition of $l = l(n)$). Next, using the inequality

$$\begin{aligned} T_{\langle n/2 \rangle - 1}(1+s) &\geqslant \tfrac{1}{2}(1 + (2s)^{1/2})^{\langle n/2 \rangle - 1}, \qquad s \geqslant 0, \\ &\geqslant \exp(C_3\, n s^{1/2}), \end{aligned}$$

we obtain for $t \in K_n$, and by the choice of Δ,

$$\psi_n(t) \geqslant \exp(C_4 \varepsilon_0^{1/(1+\eta)} n T(a_n)^{-\Delta}).$$

Furthermore for $t \in K_n$, (4.2) yields

$$\begin{aligned} W(t) &\geqslant \exp(-Q(a_{3l/4})) \geqslant \exp(-C_5 l) \\ &= \exp(-C_5 \varepsilon_0 n T(a_n)^{-\Delta}). \end{aligned}$$

Then for $t \in K_n$,

$$\begin{aligned} \psi_n(t)\, W^2(t) &\geqslant \exp(n T(a_n)^{-\Delta} \{ C_4 \varepsilon_0^{1/(1+\eta)} - 2C_5 \varepsilon_0 \}) \\ &\geqslant \exp(C_6 n (\log n)^{-2\Delta}), \end{aligned}$$

where $C_6 = C_6(\varepsilon_0)$, if ε_0 is small enough. Then as the length of K_n is

$$\begin{aligned} a_{3l/4} - a_{2l/3} &\geqslant C_7 a_{3l/4} T(a_{3l/4})^{-1} \qquad \text{(by (4.12))} \\ &\geqslant C_8 (\log n)^{-2}, \end{aligned}$$

$n \geqslant n_1$, (6.17) contradicts (6.16). ∎

From Lemmas 6.3 and 6.4, we deduce

THEOREM 6.5. *Let* $W \in \mathscr{W}_1$ *and* $l := l(n)$, $n \geqslant 1$, *be defined by* (6.14) *to* (6.15). *Then* $\exists\, n_1$ *and* $C_1 > 0$ *such that for* $n \geqslant n_1$, *each interval of length* $\geqslant C_1 a_n/n$ *in* $[-a_{l(n)}, a_{l(n)}]$ *contains at least one zero of* p_n.

We now estimate $(|Q'(x)|+1)/(|Q'(x_{jn})|+1)$ in $[x_{j+1,n}, x_{jn}]$:

LEMMA 6.6. *Let* $W \in \mathscr{W}_1$. *There exist* n_1 *and* C *such that uniformly for* $1 \leqslant j \leqslant n$ *and* $n \geqslant n_1$,

$$\max_{x \in [x_{j+1,n}, x_{j-1,n}]} \left\{ \left| \frac{Q'(x)| + 1}{Q'(x_{jn})| + 1} \right| \right\}^{\pm 1} \leqslant C(\log(2 + |Q'(x)|))^{1+2\Delta}, \qquad (6.18)$$

where we set $x_{0n} := a_{8n}$ *and* $x_{n+1,n} := -a_{8n}$, *and* Δ *is given by* (6.14).

Proof. Let $l = l(n)$ be given by (6.15), $n \geqslant 1$. Let $I_{jn} := [x_{j+1,n}, x_{j-1,n}]$. If first $I_{jn} \subset [-2, 2]$, then (6.18) follows directly. Suppose next $I_{jn} \subset [-a_l, -1] \cup [1, a_l]$. Then

$$\begin{aligned} \left| \log \left[\frac{Q'(x)^2 + 1}{Q'(x_{jn})^2 + 1} \right] \right| &= \left| \int_{x_{jn}}^{x} \frac{2Q''(t)\, Q'(t)}{Q'(t)^2 + 1}\, dt \right| \\ &\leqslant 2(x_{j-1,n} - x_{j+1,n}) \max_{t \in [x_{j+1,n}, x_{j-1,n}]} |Q''(t)/Q'(t)| \\ &\leqslant C_1 \frac{a_n}{n} \max_{t \in [x_{j+1,n}, x_{j-1,n}]} |(T(t) - 1)/t| \\ &\leqslant C_2 \frac{a_n}{n} T(a_{5n}) \leqslant C_3 \frac{a_n}{n} (\log n)^2 = o(1), \end{aligned}$$

by (4.9) and (4.5). Here we have also used Theorem 6.5 to bound $x_{j-1,n} - x_{j+1,n}$ for $n \geqslant n_1$. Together with our considerations about $[-2, 2]$, this yields (6.18) for $n \geqslant n_1$ whenever $I_{jn} \subset [-a_l, a_l]$.

Next, we note that for $n \geqslant n_1$, I_{jn} cannot contain the interval $[a_{l/2}, a_l]$ or $[-a_l, -a_{l/2}]$, since

$$\begin{aligned} a_l - a_{l/2} &\geqslant C_4 a_l T(a_l)^{-1} \qquad \text{(by (4.12))} \\ &\geqslant C_5 (\log n)^{-2} \geqslant 5C_1 a_n/n, \end{aligned}$$

by (4.9) and (4.5), where C_1 is as in Theorem 6.5: So $[a_{l/2}, a_l]$ certainly contains more than 5 zeros of p_n for n large enough. Thus if I_{jn} is not contained in $[-a_l, a_l]$, then for $n \geqslant n_1$, either

$$I_{jn} \subset [a_{l/2}, \infty) \qquad \text{or} \qquad I_{jn} \subset (-\infty, -a_{l/2}].$$

Suppose the former. Then since $x_{1n} \leqslant a_{5n}$ and $x_{0n} = a_{8n}$,

$$|Q'(x_{jn})| \leqslant Q'(a_{8n}) \leqslant C_6 n(\log n)/a_n,$$

by (4.7). Further, for $x \in I_{jn}$, (4.7) shows that

$$|Q'(x)| \geqslant Q'(a_{l/2}) \geqslant (l/2)/a_{l/2}$$

so

$$\begin{aligned}\frac{|Q'(x_{jn})|+1}{|Q'(x)|+1} &\leqslant C_7 \frac{n}{l} (\log n) \frac{a_{l/2}}{a_n} \leqslant C_7 \frac{n}{l} (\log n) \\ &\leqslant C_8 T(a_n)^{\Delta}(\log n) \qquad \text{(by (6.15))} \\ &\leqslant C_9 (\log n)^{1+2\Delta},\end{aligned}$$

by (4.9). Furthermore, in view of (4.8),

$$\log Q'(x) \geqslant \log Q'(a_{l/2}) \geqslant C_{10} \log l \geqslant C_{11} \log n.$$

So

$$\frac{|Q'(x_{jn})|+1}{|Q'(x)|+1} \leqslant C_{12}(\log(2+Q'(x)))^{1+2\Delta}.$$

If we reverse the roles of x_{jn} and x, we obtain the same bound for the reciprocal of the last left hand side. Similarly if $I_{jn} \subset (-\infty, -a_{l/2}]$. ∎

By very similar, but easier means, we can prove

LEMMA 6.7. *Let $W \in \mathscr{W}_1$. There exist n_1 and C such that uniformly for $1 \leqslant j \leqslant n$ and $n \geqslant n_1$,*

$$\max_{x \in [x_{j+1,n}, x_{j-1,n}]} \left\{ \frac{|x|+1}{|x_{jn}|+1} \right\}^{\pm 1} \leqslant C(\log(2+|Q'(x)|))^{1/\eta}. \tag{6.19}$$

Here $x_{0n} := a_{8n}$ and $x_{n+1,n} := -a_{8n}$.

We remark that at least when T is bounded, the powers of $\log(2+|Q'(x)|)$ can be removed from (6.18) and (6.19). In fact, even when T is unbounded, only slightly stronger regularity assumptions still allow us to remove these powers. Next, we bound the Christoffel numbers.

LEMMA 6.8. *Let $W \in \mathscr{W}_1$ and β be given by* (6.1). *Then for $n \geqslant 1$ and some $C > 0$,*

$$\max_{1 \leqslant j \leqslant n} \lambda_{jn} W^{-2}(x_{jn})[1+|Q'(x_{jn})|]^{-1}[1+|x_{jn}|]^{-1}[\log(2+|Q'(x_{jn})|)]^{-2(1+\beta)}$$
$$\leqslant C a_n/n. \tag{6.20}$$

Proof. From Lemma 6.1, with $m = m(n)$ there

$$|x_{jn}| \leqslant a_m$$

implies

$$\lambda_{jn} W^{-2}(x_{jn}) \leqslant C_1 \frac{a_n}{n}$$
$$\leqslant C_2 \frac{a_n}{n} [1 + |Q'(x_{jn})][1 + |x_{jn}|][\log(2 + |Q'(x_{jn})|)]^{2(1+\beta)}. \tag{6.21}$$

Next, if $|x_{jn}| \geqslant a_m$, then

$$\begin{aligned} |Q'(x_{jn})| &\geqslant Q'(a_m) \geqslant m/a_m && \text{(by (4.7))} \\ &\geqslant \varepsilon n T(a_n)^{-\beta}/a_n && \text{(by (6.2))} \\ &\geqslant C_1 n/(a_n (\log n)^{2\beta}), \end{aligned} \tag{6.22}$$

by (4.9). Then

$$\log(2 + |Q'(x_{jn})|) \geqslant C_2 \log n,$$

$n \geqslant n_1$, and by (4.3),

$$\log(2 + |Q'(x_{jn})|) \geqslant C_3 \log(2 + x_{jn}^2).$$

Then for such j, Lemma 6.2, and then (6.22), yield

$$\lambda_{jn} W^{-2}(x_{jn}) \leqslant C_4 (2 + x_{jn}^2)^{1/2} (\log(2 + x_{jn}^2))^2$$
$$\leqslant C_5 \frac{a_n}{n} |Q'(x_{jn})| \, (\log(2 + |Q'(x_{jn})|))^{2\beta} (2 + x_{jn}^2)^{1/2} (\log(2 + x_{jn}^2))^2$$
$$\leqslant C_6 \frac{a_n}{n} [1 + |Q'(x_{jn})|][\log(2 + |Q'(x_{jn})|)]^{2+2\beta} [1 + |x_{jn}|].$$

Together with (6.21), we have proved (6.20). ∎

Next, we derive rather weak lower bounds for λ_n:

LEMMA 6.9. *Let $W \in \mathcal{W}_2$ and $\sigma := \sigma(W)$. Then*

$$\sup_{x \in \mathbb{R}} \lambda_n^{-1}(W^2, x) \, W^2(x) [1 + |Q'(x)|]^{-2\sigma} [\log(2 + |Q'(x)|)]^{-2} \leqslant C_1 \frac{n}{a_n}. \tag{6.23}$$

Proof. From the relation

$$\lambda_n^{-1}(W^2, x) = \rho_n(p_n'(x)\, p_{n-1}(x) - p_{n-1}'(x)\, p_n(x)),$$

we obtain

$$\begin{aligned}
&\sup_{x\in\mathbb{R}} \lambda_n^{-1}(W^2, x)\, W^2(x)[1+|Q'(x)|]^{-2\sigma}[\log(2+|Q'(x)|)]^{-2} \\
&\quad\leqslant \rho_n \sum_{k=0}^{1} \|p'_{n-k} W[1+|Q'|]^{-\sigma}[\log(2+|Q'|)]^{-2}\|_{L_\infty(\mathbb{R})} \\
&\qquad\times \|p_{n-1+k} W[1+|Q'|]^{-\sigma}\|_{L_\infty(\mathbb{R})} \\
&\quad\leqslant a_{5n} \sum_{k=0}^{1} C_2 \frac{n}{a_n} \|p_{n-k} W[1+|Q'|]^{-\sigma}\|_{L_\infty(\mathbb{R})} \\
&\qquad\times \|p_{n-1+k} W[1+|Q'|]^{-\sigma}\|_{L_\infty(\mathbb{R})}
\end{aligned}$$

(by (4.16) and Theorem 5.1)

$$\leqslant C_3 n/a_n,$$

by (2.9) and (4.6). ▮

We remark that results in [14, 16] imply better bounds for λ_n^{-1} for Erdös weights. For Freud weights, without any additional conditions, we prove the following lemma. The method will be used elsewhere for other purposes.

LEMMA 6.10. *Let $W \in \mathscr{W}_1$ and assume that T is bounded. Then*

$$\sup_{x\in\mathbb{R}} \lambda_n^{-1}(W^2, x)\, W^2(x) \leqslant C_1 \frac{n}{a_n}. \tag{6.24}$$

Proof. The bound (6.24) is a straightforward consequence of the Markov–Bernstein inequalities in [12], as stated in (5.6) above. For any $P \in \mathscr{P}_{n-1}$, choose $\xi \in \mathbb{R}$ such that

$$|(PW)(\xi)| = \|PW\|_{L_\infty(\mathbb{R})}.$$

Let $0 < \varepsilon < 1$. Now if $|y-\xi| \leqslant \varepsilon a_n/n$, there exists z between ξ and y such that

$$\begin{aligned}
|(PW)(y)| &= |(PW)(\xi) + (PW)'(z)(y-\xi)| \\
&\geqslant |(PW)(\xi)| - |(PW)'(z)|\, \varepsilon a_n/n \\
&\geqslant \|PW\|_{L_\infty(\mathbb{R})}[1 - C\varepsilon],
\end{aligned}$$

where C is as in (5.6) and $C \neq C(n, P)$. Choosing $\varepsilon = 1/(2C)$ yields

$$|(PW)(y)| \geqslant \|PW\|_{L_\infty(\mathbb{R})}/2, \; |y - \xi| \leqslant \varepsilon a_n/n.$$

Then for $x \in \mathbb{R}$,

$$\begin{aligned}
&\int_{-\infty}^{\infty} (PW)^2(y)\, dy/(PW)^2(x) \\
&\qquad \geqslant \int_{\xi - \varepsilon a_n/n}^{\xi + \varepsilon a_n/n} \|PW\|^2_{L_\infty(\mathbb{R})}/4\, dt/(PW)^2(x) \\
&\qquad \geqslant \varepsilon a_n/(2n).
\end{aligned}$$

Thus

$$\lambda_n(W^2, x)\, W^{-2}(x) \geqslant \varepsilon a_n/(2n) \qquad \text{for all} \quad x \in \mathbb{R}. \; \blacksquare$$

We now deduce lower bounds for the spacing of the zeros:

LEMMA 6.11. *Let* $W \in \mathscr{W}_2$ *and* $\sigma := \sigma(W)$. *Then uniformly for* $2 \leqslant j \leqslant n-1$, $n \geqslant n_1$,

$$x_{j-1,n} - x_{j+1,n} \geqslant C \frac{a_n}{n} [1 + |Q'(x_{jn})|]^{-2\sigma} [\log(2 + |Q'(x_{jn})|)]^{-2}. \tag{6.25}$$

If T is bounded, we have uniformly for $2 \leqslant j \leqslant n-1$, $n \geqslant n_1$,

$$x_{j-1,n} - x_{j+1,n} \geqslant C \frac{a_n}{n}. \tag{6.26}$$

Proof. We use the Markov–Posse–Stieltjes inequality in the form given in [11, p. 89, Lemma 3.2]. Suppose first $x_{j+1,n} > 0$ and G is the entire function of Lemma 3.2 above. Setting $x_{0n} := \infty$, we have by [11, p. 89],

$$\begin{aligned}
\lambda_{jn} G(x_{jn}) &= \frac{1}{2} \left[\sum_{k\,:\,|x_{kn}| < x_{j-1,n}} \lambda_{kn} G(x_{kn}) - \sum_{k\,:\,|x_{kn}| < x_{jn}} \lambda_{kn} G(x_{kn}) \right] \\
&\leqslant \frac{1}{2} \left[\int_{-x_{j-1,n}}^{x_{j-1,n}} - \int_{-x_{j+1,n}}^{x_{j+1,n}} \right] G(t)\, W^2(t)\, dt \\
&= \int_{x_{j+1,n}}^{x_{j-1,n}} G(t)\, W^2(t)\, dt.
\end{aligned}$$

By Lemma 3.2 above, we obtain

$$x_{j-1,n} - x_{j+1,n} \geqslant C_1 \lambda_{jn} W^{-2}(x_{jn}).$$

Then (6.25) and (6.26) follow respectively from the bounds of Lemma 6.9 and 6.10. By symmetry, the same inequality holds if $x_{j-1,n} < 0$.

Finally, suppose $x_{j+1,n} \leqslant 0$ and $x_{j-1,n} \geqslant 0$. Then in view of Theorem 6.5, for $n \geqslant n_1$, both are contained in $[-1, 1]$. By the classical Markov–Stieltjes inequality,

$$\lambda_{jn} \leqslant \int_{x_{j+1,n}}^{x_{j-1,n}} W^2(t)\, dt \leqslant C_3(x_{j-1,n} - x_{j+1,n}).$$

Applying Lemmas 6.9 and 6.10 again, and the fact that W^{-2} is bounded in $[-1, 1]$, we obtain (6.25) and (6.26) for all $2 \leqslant j \leqslant n-1$. ∎

7. Proofs of the Theorems

In this section, we prove slight improvements of Theorem 2.3 and its corollaries.

Lemma 7.1. *Let $W \in \mathscr{W}_2$ and $\sigma := \sigma(W)$. Let $f: \mathbb{R} \to \mathbb{R}$ satisfy*

$$A_1 := \sup_{x \in \mathbb{R}} |f(x)|\, W^2(x)(1 + |Q'(x)|)^{2\sigma+1+\varepsilon}(1 + |x|) < \infty, \tag{7.1}$$

for some $\varepsilon > 0$. Let

$$v_n(x) := \frac{a_n^2}{n} \sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn}) \left(\frac{p_n(x)}{x - x_{jn}}\right)^2, \tag{7.2}$$

$n \geqslant 1$, $x \in \mathbb{R}$. Then for $x \in \mathbb{R}$ and $n \geqslant 1$,

$$|Y_n(W^2, f, x)| \leqslant C A_1 v_n(x), \tag{7.3}$$

where $C \neq C(n, f, x)$ but $C = C(W, \varepsilon)$.

Proof. First note that from the representation (1.27), and from (2.9), (4.16), and (4.6), for $x \in \mathbb{R}$, $n \geqslant 1$, and $1 \leqslant j \leqslant n$,

$$|l_{jn}(x)| \leqslant C_1 \lambda_{jn} a_n^{1/2} W^{-1}(x_{jn})[1 + |Q'(x_{jn})|]^{\sigma} \left|\frac{p_n(x)}{x - x_{jn}}\right|. \tag{7.4}$$

Then

$$\begin{aligned} |Y_n(W^2, f, x)| &\leqslant \sum_{j=1}^{n} |f(x_{jn})|\, l_{jn}^2(x) \\ &\leqslant C_1^2 a_n \sum_{j=1}^{n} \lambda_{jn}^2 W^{-2}(x_{jn})\, |f(x_{jn})|\, [1 + |Q'(x_{jn})|]^{2\sigma} \left(\frac{p_n(x)}{x - x_{jn}}\right)^2. \end{aligned} \tag{7.5}$$

Now by Lemma 6.8 and (7.1),

$$\begin{aligned}
&\lambda_{jn}\,|f(x_{jn})|\,[1+|Q'(x_{jn})|]^{2\sigma}\\
&\quad\leqslant C_2\frac{a_n}{n}\,|f(x_{jn})|\,W^2(x_{jn})[1+|Q'(x_{jn})|]^{2\sigma+1}\\
&\qquad\times[1+|x_{jn}|][\log(2+|Q'(x_{jn})|)]^{2(1+\beta)}\\
&\quad\leqslant C_3\frac{a_n}{n}A_1.
\end{aligned}$$

Substituting into (7.5) yields (7.3). ∎

Next, we estimate a term that enables us to compare H_n, $\hat{H}_n$, and H_n^* to Y_n:

LEMMA 7.2. *Let $W\in\mathscr{W}_2$ and $\sigma:=\sigma(W)$. Let $\{e_{jn}\}_{j,n}$ satisfy*

$$B_1:=\sup_{\substack{1\leqslant j\leqslant n\\ n\geqslant 1}}|e_{jn}|\,W^2(x_{jn})(1+|Q'(x_{jn})|)^{2\sigma+1+\delta}(1+|x_{jn}|)^2<\infty,\qquad(7.6)$$

some $\delta>0$. Then for $x\in\mathbb{R}$ and $n\geqslant 1$,

$$\begin{aligned}
&(1+|Q'(x)|)^{-\sigma}\sum_{j=1}^{n}|e_{jn}|\,|x-x_{jn}|\,l_{jn}^2(x)\,W(x)\\
&\quad\leqslant CB_1\left(\frac{a_n}{n}v_n(x)\right)^{1/2},\qquad(7.7)
\end{aligned}$$

where $C\neq C(n,\{e_{jn}\},x)$ but $C=C(W,\delta)$ and where $v_n(x)$ is defined by (7.2).

Proof. By the Cauchy–Schwarz inequality,

$$\begin{aligned}
&\sum_{j=1}^{n}|e_{jn}|\,|x-x_{jn}|\,l_{jn}^2(x)\\
&\quad\leqslant\left(\sum_{j=1}^{n}|e_{jn}|\,(x-x_{jn})^2 l_{jn}^2(x)\right)^{1/2}\left(\sum_{j=1}^{n}|e_{jn}|\,l_{jn}^2(x)\right)^{1/2}.\qquad(7.8)
\end{aligned}$$

If we define $f_n:\mathbb{R}\to\mathbb{R}$ by

$$f_n(x_{jn}):=|e_{jn}|,\qquad 1\leqslant j\leqslant n \text{ and } n\geqslant 1,$$

and $f_n(x)=0$ otherwise, then the exact argument of the previous lemma shows that

$$\sum_{j=1}^{n} |e_{jn}|\, l_{jn}^2(x) = Y_n(W^2, f_n, x)$$

$$\leqslant C_1 \{ \sup_{\substack{1 \leqslant j \leqslant n \\ n \geqslant 1}} |e_{jn}|\, W^2(x_{jn})(1+|Q'(x_{jn})|)^{2\sigma+1+\delta}(1+|x_{jn}|)\}\, v_n(x)$$

$$\leqslant C_1 B_1 v_n(x), \tag{7.9}$$

where $C_1 \neq C_1(n, \{e_{jn}\}, x)$. Furthermore, by (2.9),

$$\sum_{j=1}^{n} |e_{jn}|\, (x-x_{jn})^2 l_{jn}^2(x)\, W^2(x)$$

$$= \rho_n^2 (p_n W)^2(x) \sum_{j=1}^{n} |e_{jn}|\, \lambda_{jn}^2\, p_{n-1}^2(x_{jn})$$

$$\leqslant C_2 (1+|Q'(x)|)^{2\sigma} \sum_{j=1}^{n} |e_{jn}|\, \lambda_{jn}^2 (1+|Q'(x_{jn})|)^{2\sigma} W^{-2}(x_{jn})$$

(by (2.9), (4.16) and (4.6))

$$\leqslant C_3 (1+|Q'(x)|)^{2\sigma} \frac{a_n}{n}$$

$$\times \sum_{j=1}^{n} |e_{jn}|\, \lambda_{jn} (1+|Q'(x_{jn})|)^{2\sigma+1}$$

$$\times (1+|x_{jn}|)(\log(2+|Q'(x_{jn})|))^{2(1+\beta)}$$

(by Lemma 6.8)

$$\leqslant C_4 (1+|Q'(x)|)^{2\sigma} \frac{a_n}{n} B_1 \sum_{j=1}^{n} \lambda_{jn} W^{-2}(x_{jn})$$

$$\times (1+|Q'(x_{jn})|)^{-\delta/2}(1+|x_{jn}|)^{-1}$$

(for $n \geqslant n_1$ and by (7.6))

$$\leqslant C_5 (1+|Q'(x)|)^{2\sigma} \frac{a_n}{n} B_1, \tag{7.10}$$

by Lemma 6.2, and since for some $\alpha > 0$ and $n \geqslant n_1$,

$$(1+|Q'(x_{jn})|)^{\delta/2} \geqslant C_6 (1+|x_{jn}|)^{\alpha} \geqslant C_7 (\log(2+x_{jn}^2))^2,$$

by (4.3). Substituting (7.9) and (7.10) into (7.8) yields the lemma. ∎

With the aid of Lemma 7.2, we shall prove:

LEMMA 7.3. *Let* $W \in \mathscr{W}_2$ *and* $\sigma := \sigma(W)$. *Let* $f: \mathbb{R} \to \mathbb{R}$, *and assume that* A, *defined by* (2.13), *is finite for some* $\varepsilon > 0$.

(a) *Then for* $x \in \mathbb{R}$ *and* $n \geq 1$,

$$|Y_n(W^2, f, x) - H_n(W^2, f, x)| \; W(x)(1 + |Q'(x)|)^{-\sigma} \leq C_1 A \left[\frac{a_n}{n} v_n(x)\right]^{1/2}. \tag{7.11}$$

(b) *Suppose that* $\{d_{jn}\}_{j,n}$ *satisfy for some* $\delta > 0$, *that* B *of* (2.18) *is finite. Then for* $x \in \mathbb{R}$ *and* $n \geq 1$,

$$|Y_n(W^2, f, x) - H_n^*(W^2, f, \{d_{ln}\}, x)| \; W(x)[1 + |Q'(x)|]^{-\sigma} \leq C_2[A + B] \left[\frac{a_n}{n} v_n(x)\right]^{1/2}. \tag{7.12}$$

(c) *If* f' *exists in* $\mathbb{R}$ *and* D, *defined by* (2.20), *is finite, then for* $x \in \mathbb{R}$ *and* $n \geq 1$,

$$|Y_n(W^2, f, x) - \hat{H}_n(W^2, f, x)| \; W(x)[1 + |Q'(x)|]^{-\sigma} \leq C_3[A + D] \left[\frac{a_n}{n} v_n(x)\right]^{1/2}. \tag{7.13}$$

The constants C_1, C_2, C_3 *are independent of* n, f, x, *and* $\{d_{ln}\}$ *but depend on* ε, δ, *and* W.

Proof. (a) From (1.7), (1.9), and (1.14), we see that

$$\begin{aligned}|Y_n(W^2, f, x) - H_n(W^2, f, x)| &= \left|\sum_{j=1}^{n} f(x_{jn}) \frac{p_n''(x_{jn})}{p_n'(x_{jn})} (x - x_{jn}) \, l_{jn}^2(x)\right| \\ &\leq C_4 \sum_{j=1}^{n} |f(x_{jn})| \, [1 + |Q'(x_{jn})|] \, |x - x_{jn}| \, l_{jn}^2(x),\end{aligned}$$

by (4.18). Applying Lemma 7.2 with

$$e_{jn} := |f(x_{jn})| \, [1 + |Q'(x_{jn})|] \qquad \forall j, n$$

yields (7.11).

(b) Now by (1.9) and (1.12),

$$\begin{aligned}&|Y_n(W^2, f, x) - H_n^*(W^2, f, \{d_{ln}\}, x)| \\ &\quad\leq |Y_n(W^2, f, x) - H_n(W^2, f, x)| + \left|\sum_{j=1}^{n} d_{jn}(x - x_{jn}) \, l_{jn}^2(x)\right|.\end{aligned}$$

Applying (a) and Lemma 7.2 with $e_{jn} := d_{jn} \; \forall j, n$, yields (7.12).

(c) By applying (b) to the special case $d_{jn} = f'(x_{jn})$, we immediately obtain (7.13). ∎

It remains to estimate $v_n(x)$ before proving the boundedness and convergence results.

THEOREM 7.4. *Let $W \in W_2$ and $\sigma := \sigma(W)$. Let $v_n(x)$ be defined by* (7.2). *Then for $x \in \mathbb{R}$ and $n \geqslant 1$,*

$$v_n(x)\, W^2(x) \leqslant C_1[1+|Q'(x)|]^{\max\{2\sigma+1, 4\sigma\}}[1+|x|][\log(2+|Q'(x)|)]^{C_2}, \tag{7.14}$$

where C_1, C_2 are independent of n and x. If T is bounded, we may replace $\max\{2\sigma+1, 4\sigma\}$ *by* $2\sigma+1$.

Proof. Because of the symmetry of the zeros of p_n, it is not difficult to see from (7.2) that v_n is even. So we treat only $x \in [0, \infty)$. Set $x_{0n} := \infty$ and choose $k \geqslant 0$ such that

$$x_{k+1,n} \leqslant x < x_{kn}.$$

By (3.6) of Theorem 3.1, at least if $k \geqslant 4$,

$$\sum_{j:\,|x_{jn}| \geqslant x_{k-3,n}} \lambda_{jn} W^{-2}(x_{jn})\, |x-x_{jn}|^{-2} \leqslant C_3(x_{k-2,n}-x)^{-1}.$$

Of course if $k \leqslant 3$, this sum is taken as 0. Now by Lemma 6.11,

$$\begin{aligned} x_{k-2,n}-x &\geqslant x_{k-2,n}-x_{kn} \\ &\geqslant C_4 \frac{a_n}{n}[1+|Q'(x_{k-1,n})|]^{-2\sigma}[\log(2+|Q'(x_{k-1,n})|)]^{-2} \\ &\geqslant C_5 \frac{a_n}{n}[1+|Q'(x)|]^{-2\sigma}[\log(2+|Q'(x)|)]^{-C_6}, \end{aligned}$$

by two applications of Lemma 6.6. If T is bounded, we may omit the factors involving $Q'(x)$. Next, by (3.5) of Theorem 3.1, at least if $x_{k+4,n} \geqslant 0$,

$$\sum_{j:\,|x_{jn}| \leqslant x_{k+4,n}} \lambda_{jn} W^{-2}(x_{jn})\, |x-x_{jn}|^{-2} \leqslant C_6(x-x_{k+3,n})^{-1}.$$

If $x_{k+4,n} < 0$, this sum is taken as 0. Now by Lemma 6.11,

$$\begin{aligned} x-x_{k+3,n} &\geqslant x_{k+1,n}-x_{k+3,n} \\ &\geqslant C_7 \frac{a_n}{n}[1+|Q'(x_{k+2,n})|]^{-2\sigma}[\log(2+|Q'(x_{k+2,n})|)]^{-2} \\ &\geqslant C_7 \frac{a_n}{n}[1+|Q'(x)|]^{-2\sigma}[\log(2+|Q'(x)|)]^{-2}, \end{aligned}$$

as $x > x_{k+2,n} \geqslant 0$. If T is bounded, we may omit the factors involving Q'.

Using our bound (2.9) yields

$$\frac{a_n^2}{n}(p_n W)^2(x)\left\{\sum_{j:\,|x_{jn}|\leqslant x_{k+4,n}} + \sum_{j:\,|x_{jn}|\geqslant x_{k-3,n}}\right\}\lambda_{jn} W^{-2}(x_{jn})\,|x-x_{jn}|^{-2}$$
$$\leqslant C_8(1+|Q'(x)|)^{4\sigma}[\log(2+|Q'(x)|)]^{C_9}, \tag{7.15}$$

for $x\in\mathbb{R}$ and $n\geqslant 1$. Either sum is omitted, if empty. If T is bounded, we may replace 4σ by 2σ. It remains to estimate

$$\sum\nolimits_1 := \frac{a_n^2}{n}\sum_{j:\,x_{k+3,n}\leqslant |x_{jn}|\leqslant x_{k-2,n}} \lambda_{jn} W^{-2}(x_{jn})\left(\frac{(p_n W)(x)}{x\times x_{jn}}\right)^2$$
$$\leqslant 2\,\frac{a_n^2}{n}\sum_{j=\max\{k-2,\,1\}}^{\min\{k+3,\,n\}} \lambda_{jn} W^{-2}(x_{jn})\left(\frac{(p_n W)(x)}{x-x_{jn}}\right)^2. \tag{7.16}$$

Now by Lemma 6.8, for $\max\{k-2, 1\}\leqslant j\leqslant \min\{k-3, n\}$,

$$\lambda_{jn} W^{-2}(x_{jn})\leqslant C_{10}\,\frac{a_n}{n}\,[1+|Q'(x_{jn})|][1+|x_{jn}|][\log(2+|Q'(x_{jn})|)]^{2(1+\beta)}$$
$$\leqslant C_{11}\,\frac{a_n}{n}\,[1+|Q'(x)|][1+|x|][\log(2+|Q'(x)|)]^{C_{12}}, \tag{7.17}$$

by Lemmas 6.6 and 6.7 at least if $x\leqslant a_{8n}$. If $x\geqslant a_{8n}$, we can use the fact that $Q'(\cdot)$ is increasing in $(0, \infty)$. If, first, $x\leqslant a_{8n}$, there exists ξ between x and x_{jn} such that

$$\begin{aligned}|p_n(x)\,W(x)/(x-x_{jn})|&\\ &=|(p_n W)'(\xi)|\\ &\leqslant C_{13}\,\frac{n}{a_n}\,[\log(2+|Q'(\xi)|)]^2[1+|Q'(\xi)|]^{\sigma}\,\|p_n W[1+|Q'|]^{-\sigma}\|_{L_\infty(\mathbb{R})}\end{aligned}$$

(by Theorem 5.1)

$$\leqslant C_{14}\,\frac{n}{a_n^{3/2}}\,[1+|Q'(x)|]^{\sigma}[\log(2+|Q'(x)|)]^{C_{15}}, \tag{7.18}$$

by (2.9) and a fixed number of applications of Lemma 6.6. On the other hand if $x\geqslant a_{8n}$,

$$|(p_n W)(x)/(x-x_{jn})|\leqslant C_{16}a_n^{-1/2}(1+|Q'(x)|)^{\sigma}/(a_{8n}-a_{5n})$$

(by (2.9) and (4.17))

$$\leqslant C_{17}a_n^{-3/2}(1+|Q'(x)|)^{\sigma}(\log n)^2,$$

by (4.13) and (4.14). Hence (7.18) remains true. Substituting (7.17) and (7.18) into (7.16) yields

$$\sum\nolimits_1 \leqslant C_{18}[1+|Q'(x)|]^{2\sigma+1}[1+|x|][\log(2+|Q'(x)|)]^{C_{16}}.$$

Combined with (7.15), this yields (7.14). ∎

We can now prove:

THEOREM 7.5. *Let $W \in \mathscr{W}_2$ and $\sigma = \sigma(W)$. Let $f: \mathbb{R} \to \mathbb{R}$ and assume that A_1 defined by* (7.1) *is finite. Let κ and V be given by* (2.14a, b) *and* (2.15), *respectively. Then for $n \geqslant 1$,*

$$\| Y_n(W^2, f, \cdot)\, W^2 V \|_{L_\infty(\mathbb{R})} \leqslant C A_1, \tag{7.19}$$

where $C \neq C(n, f)$.

Proof. This follows directly from Lemma 7.1 and Theorem 7.4. ∎

THEOREM 7.6. *Let $W \in \mathscr{W}_2$ and $\sigma = \sigma(W)$. Let $f: \mathbb{R} \to \mathbb{R}$ and assume that A defined by* (2.13) *is finite. If T is unbounded, let*

$$\kappa_1 > \max\{2\sigma + \tfrac{1}{2}, 3\sigma\}, \tag{7.20a}$$

and if T is bounded, let

$$\kappa_1 > 2\sigma + \tfrac{1}{2}. \tag{7.20b}$$

Furthermore, let

$$V_1(x) := [1+|Q'(x)|]^{-\kappa_1}[1+|x|]^{-1/2}, \qquad x \in \mathbb{R}. \tag{7.21}$$

(a) *Then*

$$\|(Y_n(W^2, f, \cdot) - H_n(W^2, f, \cdot))\, W^2 V_1\|_{L_\infty(\mathbb{R})} \leqslant C_1 A (a_n/n)^{1/2}. \tag{7.22}$$

(b) *Assume that B and $\{d_{jn}\}_{j,n}$ satisfy* (2.18) *for some $\delta > 0$. Then*

$$\|(Y_n(W^2, f, \cdot) - H_n^*(W^2, f, \{d_{ln}\}, \cdot))\, W^2 V_1\|_{L_\infty(\mathbb{R})} \leqslant C_2 [A + B](a_n/n)^{1/2}. \tag{7.23}$$

(c) *Assume that f' exists in $\mathbb{R}$, and D defined by* (2.20), *is finite. Then*

$$\|(Y_n(W^2, f, \cdot) - \hat{H}_n(W^2, f, \cdot))\, W^2 V_1\|_{L_\infty(\mathbb{R})} \leqslant C_3 [A + D](a_n/n)^{1/2}. \tag{7.24}$$

Here C_1, C_2, C_3 are independent of n and f.

Proof. This follows directly from Lemma 7.3 and Theorem 7.4. ∎

Proof of Theorem 2.3. (a) From (2.13) and (7.1),

$$A_1 \leqslant A.$$

Then (7.19) in Theorem 7.5 yields (2.16).

(b) From (2.14a, b), (2.15) and (7.20a, b), (7.21), we see that

$$V \leqslant V_1,$$

provided κ_1 is chosen so close to its lower bound that $\kappa_1 \leqslant \kappa$. Then Theorem 7.6(a), (b), (c) yield respectively Theorem 2.3(b), (c), (d). ∎

To deduce convergence of the operators from Theorem 2.3, we prove convergence on the polynomials:

THEOREM 7.7. *Let $W \in \mathscr{W}_2$ and $\sigma = \sigma(W)$. Let κ_1 be given by* (7.20a, b) *and V_1 by* (7.21). *Let $\varepsilon > 0$. Then for $n \geqslant 1$ and $R_n \in \mathscr{P}_{2n-1}$,*

$$\|(H_n(W^2, R_n, \cdot) - R_n(\cdot))\, W^2 V_1\|_{L_\infty(\mathbb{R})} \leqslant C\left(\frac{a_n}{n}\right)^{1/2} \sup_{t \in \mathbb{R}} \{|R_n'(t)|\, W^2(t)[1 + |Q'(t)|]^{2\sigma+1+\varepsilon}[1 + |t|]^2\}, \tag{7.25}$$

where $C \neq C(n, R_n)$.

Proof. Now by [28, p. 44],

$$H_n(W^2, R_n, x) - R_n(x) = -\sum_{j=1}^{n} R_n'(x_{jn})(x - x_{jn})\, l_{jn}^2(x).$$

Applying Lemma 7.2 yields, for $x \in \mathbb{R}$,

$$(1 + |Q'(x)|)^{-\sigma} |\, H_n(W^2, R_n, x) - R_n(x)|\, W(x) \leqslant C_1\left(\frac{a_n}{n} v_n(x)\right)^{1/2} \sup_{t \in \mathbb{R}} \{|R_n'(t)|\, W^2(t)[1 + |Q'(t)|]^{2\sigma+1+\varepsilon}[1 + |t|]^2\}.$$

Then Theorem 7.4 yields the result. ∎

Proof of Corollary 2.4. Since $V \leqslant V_1$ with a suitable choice of κ_1, Theorem 7.7 and (4.5) yield

$$\lim_{n \to \infty} \|(H_n(W^2, R, \cdot) - R(\cdot))\, W^2 V\|_{L_\infty(\mathbb{R})} = 0,$$

$\forall$ polynomial R. Then for any such polynomial, Theorem 2.3(b) ensures that

$$\lim_{n \to \infty} \|(Y_n(W^2, R, \cdot) - R(\cdot))\, W^2 V\|_{L_\infty(\mathbb{R})} = 0.$$

In view of the fact that we can find a polynomial R such that for a given $\alpha > 0$,

$$\sup_{x \in \mathbb{R}} |f - R|\,(x)\, W^2(x)[1 + |Q'(x)|]^{2\sigma + 2 + \varepsilon}[1 + |x|]^2 < \alpha$$

(cf. [5, p. 180]), then by Theorem 2.3(a),

$$\begin{aligned}
&\|(Y_n(W^2, f, \cdot) - f(\cdot))\, W^2 V\|_{L_\infty(\mathbb{R})} \\
&\quad = \|\{Y_n(W^2, f - R, \cdot) - (f - R)(\cdot) + Y_n(W^2, R, \cdot) - R(\cdot)\}\, W^2 V\|_{L_\infty(\mathbb{R})} \\
&\quad \leqslant C_1 \sup_{x \in \mathbb{R}} |f - R|\,(x)\, W^2(x)[1 + |Q'(x)|]^{2\sigma + 2 + \varepsilon}[1 + |x|]^2 \\
&\qquad + \|(f - R)\, W^2 V\|_{L_\infty(\mathbb{R})} + o(1) \\
&\quad \leqslant C_2 \sup_{x \in \mathbb{R}} |f - R|\,(x)\, W^2(x)[1 + |Q'(x)|]^{2\sigma + 2 + \varepsilon}[1 + |x|]^2 + o(1) \\
&\quad \leqslant C_2 \alpha + o(1),
\end{aligned}$$

since $V \leqslant 1$. Hence (2.22). The equiconvergence results of Theorem 2.3 then imply convergence of $\{H_n\}$, $\{H_n^*\}$, and $\{\hat{H}_n\}$. ∎

Proof of Corollary 2.5. By (1.20),

$$\begin{aligned}
&|J_n[k; f] - I[k; f]| \\
&\quad = \left| \int_{-\infty}^{\infty} (Y_n(W^2, f, x) - f(x))\, k(x)\, dx \right| \\
&\quad \leqslant \|(Y_n(W^2, f, \cdot) - f(\cdot))\, W^2 V\|_{L_\infty(\mathbb{R})} \int_{-\infty}^{\infty} |k(x)|\, W^{-2}(x)\, V^{-1}(x)\, dx.
\end{aligned}$$

Now apply Corollary 2.4. The remaining results are similar. ∎

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Note added in proof. The bound (2.11) has been proved for a class of weights including $\exp(-|x|^{\alpha})$, $\alpha > 1$. This is contained in "Christoffel Functions, Orthogonal Polynomials, and Nevai's Conjecture for Freud Weights," by A. L. Levin and the author, to appear in *Constructive Approximation.*

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